BENG 221: Mathematical Methods in Bioengineering

Lecture 9

Green's Functions for ODEs and PDEs

References

Haberman APDE, Sec. 8.2 and 8.3.

Haberman APDE, Sec. 9.1 and 9.2.

http://en.wikipedia.org/wiki/Green's_function

See also Lecture 8 notes.

GREEN'S FUNCTION FOR ODEs

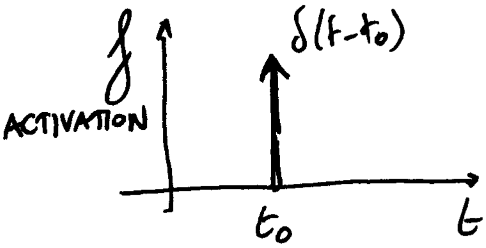
$$\underbrace{\mathcal{L} u}_{\text{RESPONSE}} = \underbrace{f}_{\text{ACTIVATION}}$$

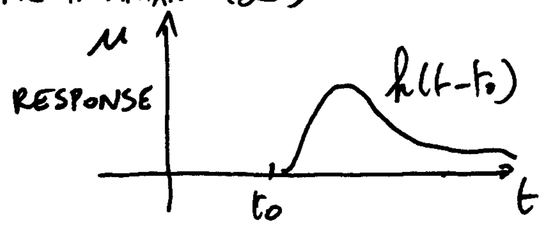
$$f = \delta(t - t_0) \xrightarrow{f(t)} \boxed{\text{LTI}} \xrightarrow{u(t)} u = h(t - t_0)$$

IMPULSE

IMPULSE RESPONSE

LINEAR TIME-INVARIANT ($\mathcal{L}$)





$$f(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow u = \int_{-\infty}^{t} f(t_0)\, h(t - t_0)\, dt_0$$

ACTIVATION

CONVOLVED RESPONSE

More generally, for a linear time-variant system:

$$f(t) \longrightarrow \boxed{\text{LTV}} \longrightarrow u = \int_{-\infty}^{t} f(t_0)\, G(t, t_0)\, dt_0$$

where $G(t, t_0)$ is the response of the system to an impulse at time t_0.

$G(t, t_0)$ is called the GREEN'S FUNCTION of the ODE system.

With initial conditions: (when $u(t) = 0$ for $t < 0$)

$$\frac{du}{dt} = \mathcal{L}(u(t)) + f(t) \quad \text{with I.C.} \quad u(0) = u_0$$

Equivalently:

$$\frac{du}{dt} = \mathcal{L}(u(t)) + f(t) + u_0\,\delta(t) \quad \text{with zero I.C.}$$

Equivalent because: $\left(\begin{array}{l} 0^- = -\varepsilon \\ 0^+ = +\varepsilon \end{array} \text{ for } \lim_{\varepsilon \to 0} \right)$

$$\underbrace{\int_{0^-}^{0^+} \frac{du}{dt} \cdot dt}_{u(0^+) - u(0^-)} = \underbrace{\int_{0^-}^{0^+} \left(\mathcal{L}(u(t)) + f(t)\right) dt}_{=0} + u_0 \underbrace{\int_{0^-}^{0^+} \delta(x)\, dt}_{1 \text{ by definition}}$$

or: $u(0^+) = u(0^-) + u_0 = u_0$

(i.e., u steps from 0 to u_0 at $t = 0$)

$$\Rightarrow u(t) = \int_{-\infty}^{t} \left(u_0\,\delta(t_0) + f(t_0)\right) G(t, t_0)\, dt_0 \quad \text{where } f(t_0) = 0 \text{ for } t_0 < 0$$

$$= \underbrace{u_0}_{\text{I.C.}} \; \underbrace{G(t, 0)}_{\text{GREEN'S FUNCTION @ } t_0 = 0} + \int_0^t \underbrace{f(t_0)}_{\text{DRIVING SOURCE}} \; \underbrace{G(t, t_0)}_{\text{GREEN'S FUNCTION @ } t_0} dt_0$$

Example: $\mathcal{L}(u(t)) = A \cdot u(t)$ (LTI system)

a. Homogeneous problem: $f(t) = 0$

$$\frac{du}{dt} = A\, u(t) \implies u(t) = \underbrace{c}_{\substack{\text{ARBITRARY} \\ \text{COEFFICIENT} \\ \text{(depends on I.C.)}}} \cdot \underbrace{e^{At}}_{\text{EIGEN MODE}}$$

b. Inhomogeneous problem: $f(t) \neq 0$

$$\frac{du}{dt} = A\, u(t) + f(t) \implies u(t) = \underbrace{c(t)}_{\substack{\text{VARIATION OF} \\ \text{COEFFICIENT}}} \cdot e^{At}$$

(similar to integrating factors)

$$\frac{d}{dt}\left(c(t) \cdot e^{At}\right) = \frac{dc}{dt} e^{At} + \cancel{c(t) A e^{At}} = A\, \cancel{c(t) e^{At}} + f(t)$$

$$\implies \frac{dc}{dt} = f(t)\, e^{-At} \implies c(t) - c(0) = \int_0^t f(t_0)\, e^{-At_0}\, dt_0$$

$$u(t) = \left(c(0) + \int_0^t f(t_0)\, e^{-At_0}\, dt_0 \right) e^{At}$$

I.C.: $u(0) = c(0) = u_0$

$$\implies u(t) = \underbrace{u_0}_{\text{I.C.}}\ \underbrace{e^{At}}_{G(t,0) = h(t)} + \int_0^t \underbrace{f(t_0)}_{\substack{\text{DRIVING} \\ \text{SOURCE}}} \cdot \underbrace{e^{A(t-t_0)}}_{G(t,t_0) = h(t-t_0)}\, dt_0$$

with $h(t) = e^{At}$ IMPULSE RESPONSE : solution to $\frac{dh}{dt} = Ah + \delta(t)$ from zero I.C.

(Green's function for LTI system)

GREEN'S FUNCTION FOR PDEs ON BOUNDED DOMAINS

1) HOMOGENEOUS PDEs (1 dimension)

e.g.: $$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{cases} u(0,t) = 0 & \text{B.C} \\ u(L,t) = 0 & \text{B.C.} \\ u(x,0) = g(x) & \text{I.C.} \end{cases}$$

Separation of variables:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \text{with} \quad a_n = \frac{2}{L}\int_0^L g(x_0) \sin\frac{n\pi x_0}{L} dx_0$$

Substitution gives:

$$u(x,t) = \int_0^L g(x_0) \Big(\underbrace{\sum_{n=1}^{\infty} \frac{2}{L} \sin\frac{n\pi x_0}{L} \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}}_{G(x,t;x_0)} \Big) dx_0$$

More generally: I.C. $u(x,t_0) = g(x)$ at time t_0

$$\Rightarrow u(x,t) = \int_0^L g(x_0)\, G(x,t;x_0,t_0)\, dx_0$$

$$\text{with } G(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{2}{L} \sin\frac{n\pi x_0}{L} \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-t_0)}$$

$$= G(x, t-t_0; x_0, 0) \qquad \text{(LTI PDE)}$$

2) INHOMOGENEOUS PDEs WITH HOMOGENEOUS B.C.

e.g.: $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} + \underset{\substack{\uparrow \\ \text{DRIVING} \\ \text{SOURCE}}}{Q(x,t)}$ with $\begin{cases} u(0,t)=0 & \text{B.C.} \\ u(L,t)=0 & \text{B.C.} \\ u(x,0)=g(x) & \text{I.C.} \end{cases}$

Haberman Sec. 9.2 pp 380-383, or:

consider $Q(x,t)$ as a continuous distribution of I.C.s over time $\left(\int_0^\infty dt_0 (\dots) \right)$

$$\Rightarrow u(x,t) = \int_0^L \underset{\substack{\downarrow \\ \text{I.C.}}}{g(x_0)} \; \underset{\substack{\downarrow \\ \text{GREEN'S FUNCTION} \\ \text{AT } t_0=0}}{G(x,t;x_0,0)} \, dx_0$$

$$+ \int_0^L \int_0^t \underset{\substack{\downarrow \\ \text{DRIVING} \\ \text{SOURCE}}}{Q(x_0,t_0)} \; \underset{\substack{\downarrow \\ \text{GREEN'S FUNCTION}}}{G(x,t;x_0,t_0)} \, dx_0 \, dt_0$$

Proof: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + Q(x,t)$ with $\begin{cases} u(x,0) = g(x) & \text{I.C.} \\ u(0,t) = 0 & \text{B.C. @ } 0 \\ u(L,t) = 0 & \text{B.C. @ } L \end{cases}$

(Sec 9.2 pp 381–383)

a. Homogeneous problem: $Q(x,t) = 0$

Separation of variables: $$u(x,t) = \sum_{n=1}^{\infty} \underset{\text{COEFFICIENT}}{\underset{\downarrow}{a_n}} \; \underbrace{\sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t}}_{\text{EIGENMODES}}$$

b. Inhomogeneous problem: $Q(x,t) \neq 0$

Variation of coefficients: $$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n}{dt} \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \cancel{\sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)\left(-k\left(\frac{n\pi}{L}\right)^2\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$$= k\frac{\partial^2 u}{\partial x^2} + Q = \cancel{k\sum_{n=1}^{\infty} a_n(t)\left(-\left(\frac{n\pi}{L}\right)^2\right) \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}} + Q(x,t)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{da_n}{dt} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right) \sin\left(\frac{n\pi x}{L}\right) = Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$q_n(t) = \frac{2}{L}\int_0^L Q(x_0,t) \sin\left(\frac{n\pi x_0}{L}\right) dx_0$$

FOURIER SERIES EXPANSION of $Q(x,t)$

$$\Rightarrow \frac{da_n}{dt} e^{-k\left(\frac{n\pi}{L}\right)^2 t} = q_n(t) \quad \Rightarrow \int_{a_n(0)}^{a_n(t)} da_n = \int_0^t q_n(t_0) e^{+k\left(\frac{n\pi}{L}\right)^2 t_0} dt_0$$

$$\Rightarrow a_n(t) = \underset{\substack{\text{FOURIER SERIES EXPANSION} \\ \text{of } g(x) \text{ I.C.}}}{\underset{\downarrow}{a_n(0)}} + \int_0^t \underset{\substack{\text{FOURIER SERIES EXPANSION of } Q(x,t) \\ \text{DRIVING SOURCE}}}{\underset{\downarrow}{q_n(t_0)}} \, e^{k\left(\frac{n\pi}{L}\right)^2 t_0} dt_0$$

I.C. : $g(x) = u(x,0) = \sum_{n=1}^{\infty} d_n(0) \sin\left(\frac{n\pi x}{L}\right)$

$$\rightarrow \quad d_n(0) = \frac{2}{L}\int_0^L g(x_0) \sin\left(\frac{n\pi x_0}{L}\right) dx_0$$

FOURIER SERIES EXPANSION of $g(x)$ I.C.

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left(\underbrace{d_n(0)}_{\frac{2}{L}\int_0^L g(x_0)\sin\left(\frac{n\pi x_0}{L}\right)dx_0} + \int_0^t \underbrace{q_n(t_0)}_{\frac{2}{L}\int_0^L Q(x_0,t_0)\sin\left(\frac{n\pi x_0}{L}\right)dx_0} e^{k\left(\frac{n\pi}{L}\right)^2 t_0} dt_0\right)\cdot \sin\left(\frac{n\pi x}{L}\right)\cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Interchanging the order of $\sum_{n=1}^{\infty}\cdots$ and $\int_0^L dx_0\cdots$ and $\int_0^t dt_0 \cdots$:

$$u(x,t) = \int_0^L \underbrace{g(x_0)}_{\text{I.C.}}\underbrace{\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}\right)}_{G(x,t;\,x_0,0)} dx_0$$

$$+ \int_0^L\int_0^t \underbrace{Q(x_0,t_0)}_{\text{DRIVING SOURCE}}\underbrace{\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)}\right)}_{G(x,t;\,x_0,t_0)} dx_0\, dt_0$$

NOTES:

- For LTI (linear time invariant) PDEs:

$$G(x,t;x_0,t_0) = G(x,t-t_0;x_0,0)$$

also, for LSI (linear space invariant) PDEs:

$$G(x,t;x_0,t_0) = G(x-x_0,t;0,t_0)$$

and thus for LSTI (linear space and time invariant) PDEs:

$$G(x,t;x_0,t_0) = G(x-x_0,t-t_0;0,0)$$

- For all linear PDEs:

$$G(x,t;x_0,t) = \delta(x-x_0)$$

because of causality!

- The Green's function for general linear PDEs of the form:

$$\mathcal{L}_{x,t}\, u(x,t) = f(x,t) \quad \text{with} \begin{cases} B.C. = h(t) \ldots \\ I.C. = g(x) \ldots \end{cases}$$

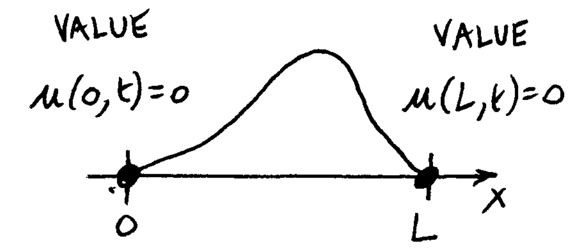
can be found as the solution to:

$$\mathcal{L}_{x,t}\, G(x,t;x_0,t_0) = \delta(x-x_0)\,\delta(t-t_0) \quad \text{with} \begin{cases} B.C = 0 \\ I.C. = 0 \end{cases}$$

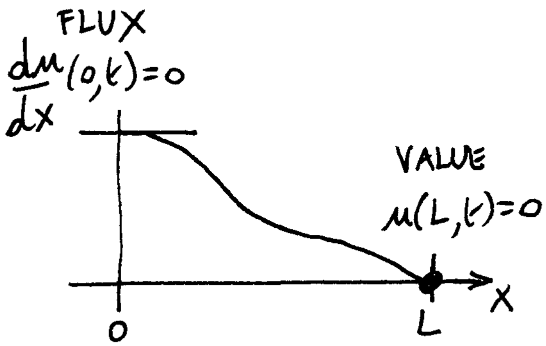
or, equivalently:

$$\mathcal{L}_{x,t}\, G(x,t;x_0,t_0) = 0 \quad \text{with} \begin{cases} B.C. = 0 \\ I.C. = \delta(x-x_0) \end{cases}$$

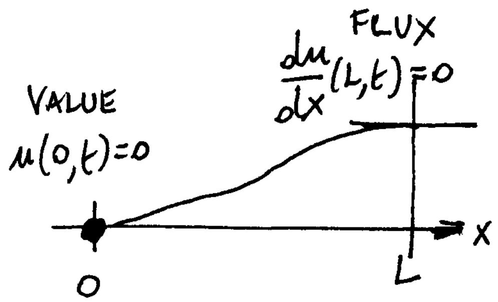
- The Green's function depends on the type of homogeneous B.C., e.g.:



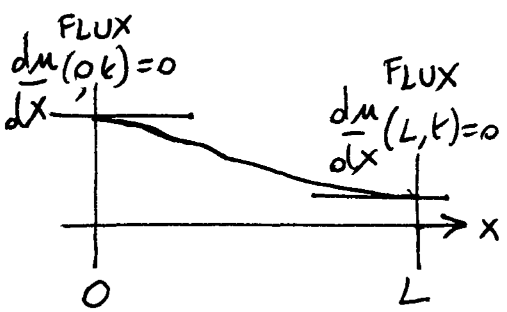
$$: G(x,t;x_0,t_0)=\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)}$$



$$: G(x,t;x_0,t_0)=\sum_{n=\underline{odd}}^{\infty}\frac{2}{L}\cos\frac{n\pi x_0}{2L}\cos\frac{n\pi x}{2L}e^{-k\left(\frac{n\pi}{2L}\right)^2(t-t_0)}\quad (n=1,3,5,\dots)$$



$$: G(x,t;x_0,t_0)=\sum_{n=\underline{odd}}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{2L}\sin\frac{n\pi x}{2L}e^{-k\left(\frac{n\pi}{2L}\right)^2(t-t_0)}\quad (n=1,3,5,\dots)$$



$$: G(x,t;x_0,t_0)=\sum_{n=1}^{\infty}\frac{2}{L}\cos\frac{n\pi x_0}{L}\cos\frac{n\pi x}{L}e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)}+\frac{1}{L}$$

Physical interpretation:

"VALUE" $u(x,t)$: TEMPERATURE ; VOLTAGE → "SHORT"

"FLUX" $k\frac{du}{dx}(x,t)$: HEAT TRANSFER ; CURRENT → "OPEN CIRCUIT"