Model of Aortic Blood Flow Using the Windkessel Effect

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October 25, 2012

Introduction

Aortic Blood Flow and the Windkessel Model

Within the human circulatory system, the aorta is the largest artery, originating from the heart's left ventricle and extending down to the abdomen, where it branches into smaller arteries.

The cardiac cycle is a closed-loop, pulsatile system: the heart pumps blood throughout the systemic circulation in a manner that resembles a pulse wave (Figure 1).
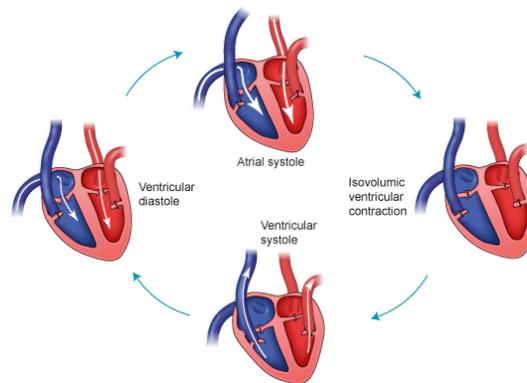

Figure 1: Cardiac cycle phases. Obtained from: `http://www.beltina.org/health-dictionary/cardiac-cycle-phases-diagram-definition.html`

The first phase of the cardiac phase, ***ventricular diastole*** happens when the ventricles are relaxed and allow for the newly oxygenated blood to flow in from the atria. Ventricular diastole is followed by systole, ***systole***, where the ventricles contract and eject the blood out to the body through the aorta. Aortic pressure rises when the ventricles contract, pumping the blood into the aorta, and, at its maximum is termed ***systolic pressure***. At the start of following cardiac cycle, as the blood begins to flow into the ventricles, the aortic pressure is at its lowest, and it is known as ***diastolic pressure***.

The ***Windkessel Model*** was designed in the late 1800's by the german physiologist Otto Frank. He described the heart and the systemic arterial system as a closed hydraulic circuit. In his analogy, the circuit contained a water pump connected to a chamber, filled with water except for a pocket of air. As it's pumped, the water compresses the air, which in turn pushes the water out of the chamber. This analogy resembles the mechanics of the heart. Windkessel models are commonly used to used to represent the load undertaken by the heart during the cardiac cycle. It relates blood pressure and blood flow in the aorta, and characterizes the arterial compliance, peripheral
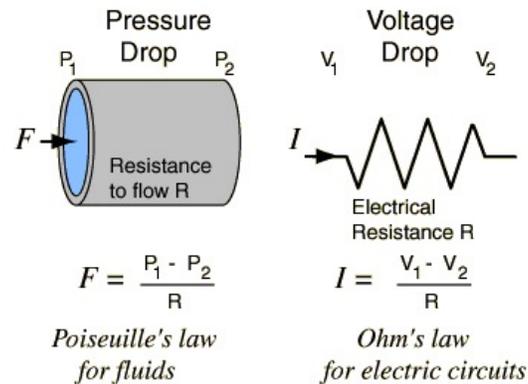
Figure 2: Fluid Dynamics and Electrical Circuit Equivalents. Obtained from: `http://hyperphysics.phy-astr.gsu.edu/hbase/electric/watcir2.html`

resistance of the valves and the inertia of the blood flow. This is relevant in the context of, for example: the effects of vasodilator or vasoconstrictor drugs, the development of mechanical hearts and heart-lung machines.

The Windkessel model takes into consideration the following parameters while modeling the cardiac cycle:

−**Arterial Compliance:** refers to the elasticity and extensibility of the major artery during the cardiac cycle.

−**Peripheral Resistance:** refers to the flow resistance encountered by the blood as it flows through the systemic arterial system.

−**Inertia:** simulates the inertia of the blood as it is cycled through the heart.

The Windkessel Model is analogous to the Poiseuille's Law for a hydraulic system. It describes the flow of blood through the arteries as the flow of fluid through pipes. In this report, we focus on the electrical circuit equivalent, as shown in Figure 2.

Problem Statement

In this project, we aim to mathematically model the blood flow to the aorta, the relationship between blood pressure and blood flow in the aorta, the compliance of blood vessels, and, to compute the analytical and numerical solutions for the same. We will also briefly critique our results and the robustness of the Windkessel Model for cardiac modeling.

Windkessel Model Description

Model Assumptions

We assume that:

 – Cardiac cycle starts at systole.

 – The period of the systole is 2/5th of the period of cardiac cycle.

 – Arterial Compliance, Peripheral Resistance, and Inertia are modeled as a capacitor, a resistor, and an inductor respectively.

The basic Windkessel model calculates the exponential pressure curve determined by the systolic and diastolic phases of the cardiac cycle. As the number of elements in the model increases, a new physiological factor is accounted for and more accurate the results are when related to the original curve. Various other criteria such as computational complexity, shape of curve generated etc. must be considered while deciding on which model to choose. These are approached in the three different Windkessel models explained below.

The 2-Element Windkessel Model

The simplest of the Windkessel models demonstrating the hemodynamic state is the *2-Element Model*. During a cardiac cycle, it takes into account the effect of arterial compliance and total peripheral resistance. In the electrical analog, the arterial compliance (C in $cm^3/mmHg$) is represented as a capacitor with electric charge storage properties; peripheral resistance of the systemic arterial system (R in $mmHg \times s/cm^3$) is represented as an energy dissipating resistor.
The flow of blood from the heart ($I(t)$ in cm^3/s) is analogous to that of current flowing in the circuit and the blood pressure in the aorta (P(t) in mmHg) is modeled as a time-varying electric potential. Figure 4 shows that during systole, there is ejection of blood from the ventricles to the compliant aortic chamber. The blood stored in peripheral vessels and the elastic recoil of aorta during diastole are depicted as solid and dashed lines respectively.
The theoretical modeling as seen in the electrical analog (Figure 3) is given as:

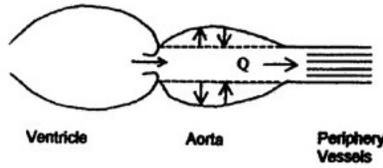$$I(t) = \frac{P(t)}{R} + C\frac{dP(t)}{dt} \tag{1}$$

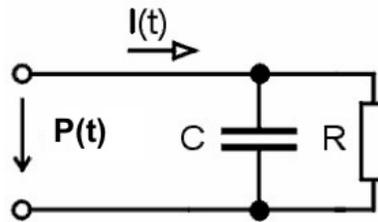Figure 3: Diagrammatic representation of ventricular ejection of blood and arterial circulation

Figure 4: Electrical Analog of the 2-Element Windkessel Model

The 3-Element Windkessel Model

The 3-Element Windkessel Model simulates the characteristic impedance of the proximal aorta. A resistor is added in series to account for this resistance to blood flow due to the aortic valve. The already existing parallel combination of resistor-capacitor represent the total peripheral resistance and aortic compliance in the 2-element model as discussed before. A hydraulic equivalent of the 3-element model is shown in Figure 5. Aortic compliance due to pressure variations is seen by allowing a bottle to undergo volume displacements. The tube geometry represents the characteristic aortic impedance. Resistance to flow is varied by partial opening and closing of needle valve shown.
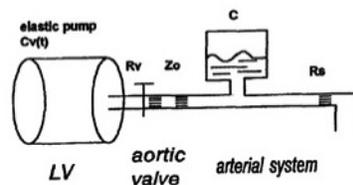

Figure 5: A Hydraulic Equivalent of the 3-Element Windkessel Model

The theoretical modeling as seen in the electrical analog (Figure 6) is given as:

$$\left(1 + \frac{r}{R}\right) i(t) + CR_1 \frac{di(t)}{dt} = \frac{P(t)}{R} + C \frac{dP(t)}{dt} \tag{2}$$
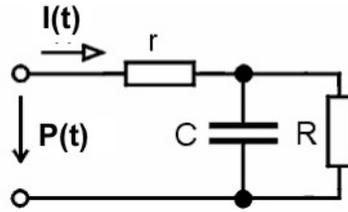
Figure 6: Electrical Analog of the 3-Element Windkessel Model

The 4-Element Windkessel Model

This model includes an inductor in the main branch of the circuit as it accounts for the inertia to blood flow in the hydrodynamic model. The drop in electrical potential across the inductor is given as L(di(t)/dt). The 4-element model gives a more accurate representation of the blood pressure vs. cardiac cycle time curve when compared to the two and the three element models. The electrical analog is shown here (Figure 7):
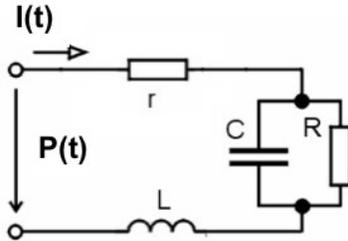

Figure 7: Electrical Analog of the 4-Element Windkessel Model

Theoretical modeling:

$$\left(1 + \frac{r}{R}\right) i(t) + \left(rC + \frac{L}{R}\right) \frac{di(t)}{dt} + LC\frac{d^2i(t)}{dt^2} = \frac{P(t)}{R} + C\frac{dP(t)}{dt} \tag{3}$$

Model of the Blood Flow to the Aorta

The flow of blood into the aorta from the ventricle during the cardiac cycle is represented as $I(t)$ in our model. $I(t)$ is modeled as a sine wave with amplitude I_0 during systole and is zero otherwise. This follows our learning of the cardiac physiology. During diastole, when the ventricles are relaxed,

there is no blood flow into the aorta, and therefore, $I(t) = 0$. However with ventricular contraction during the systole, blood is ejected into the aorta and can be modeled as a sinusoidal wave , therefore:

$$I(t) = I_0 sin\left(\pi * \frac{mod(t, T_c)}{T_s}\right) \tag{4}$$

where t is time in seconds, T_c is the period of the cardiac cycle in seconds, T_s is the period of systole, in seconds, and $mod(t, T_c)$ represents the remainder of t divided by T_c. T_s is assumed to be $2/5 T_c$, according to the dynamics of the cardiac cycle.

According to literature, the blood flow in one cardiac cycle is 90 cm^3. We use that information to obtain the constant I_0:

$$90 = \int_0^{T_c} I_0 sin\left(\pi * \frac{mod(t, T_c)}{T_s}\right) dt$$

$$I_0 = \int_0^{T_c} (1/90) sin\left(\pi * \frac{mod(t, T_c)}{T_s}\right) dt$$

$$\boxed{I_0 = 424.1 \text{ mL}}$$

Therefore the maximum amplitude of the blood flow during systole is $I_0 = 424.1$ mL

Simplified Analytical Solution

We solve analytically for the 2-Element Windkessel Model, that was given by:

$$C\frac{dP(t)}{dt} + \frac{P(t)}{R} = I(t)$$

Systolic Phase: Inhomogeneous solution

$$\frac{dP(t)}{dt} + \frac{P(t)}{CR} = I(t)$$

Using integrating factor $u(t) = \int \frac{1}{CR} dt = e^{\frac{t}{RC}}$:

$$\frac{dP(t)}{dt} e^{\frac{t}{RC}} + e^{\frac{t}{RC}} \frac{P(t)}{CR} = \frac{I_0}{C} sin\left(\pi t/T_s\right) e^{\frac{t}{RC}}$$

Note that $\dfrac{dP(t)}{dt} e^{\frac{t}{RC}} + e^{\frac{t}{RC}} \dfrac{P(t)}{CR} = \dfrac{d}{dt}\left(e^{\frac{t}{RC}} P(t)\right)$

So now we integrate both sides:

$$\int d\left(e^{\frac{t}{RC}} P(t)\right) = \int \frac{I_0}{C} sin\left(\pi t/T_s\right) e^{\frac{t}{RC}} dt$$

And our solution is:

$$\boxed{y(t) = c_1 e^{\frac{-t}{RC}} + \frac{-e^{\frac{t}{CR}} T_s I_0 R(C\pi R cos(\frac{\pi t}{T_s}) - T_s sin(\frac{\pi t}{T_s}))}{T_s^2 + C^2 \pi^2 R^2}}$$

To solve for the constant c_1, we consider the initial conditions for $P(t)$ at the start of the systolic cycle. As each systolic cycle is preceded by a diastolic cycle. At time t_{ss} = start of systolic cycle, $P(t)$ equals the diastolic pressure (P_{ss}). Therefore:

$$c_1 = P_{ss} + \frac{I_0 T_s R \left[C\pi R cos\left(\frac{\pi(t-t_{ss})}{T_s}\right) - T_s sin\left(\frac{\pi(t-t_{ss})}{T_s}\right)\right]}{T_s^2 + C^2 * \pi^2 * R^2} e^{\frac{(t-t_{ss})}{CR}}$$

Which gives us:

$$c_1 = P_{ss} + \frac{I_0 T_s R \left[C\pi R\right]}{T_s^2 + C^2 * \pi^2 * R^2}$$

Diastolic phase: homogeneous solution

$$\frac{P(t)}{R} + C\frac{dP(t)}{dt} = 0$$

$$\boxed{P(t) = ce^{\frac{-t}{RC}}}$$

To determine the constant c, we solve for the initial condition for $P(t)$ at the start of diastolic cycle. At time t_{sd} = start of diastolic cycle, $P(t)$ equals the diastolic pressure (P_{sd}). As each diastole is preceded by a systole, this is the pressure at the end of the systolic cycle. For all our analytical and numerical analysis, P_{sd} was determined from the solution of the preceding systolic cycle. We expect this number to be around 120 mmHg. This is because the blood pressure for a healthy person is around 120mmHg/80mmHg (Systolic/diastolic). Given that we started with 80mmHg as our diastolic blood pressure, our model should be able to output the systolic blood pressure as 120mmHg. However, it should be noted that the diastolic blood pressure of 80mmHg was supplied as an initial condition only for the first cycle. For the remaining cycles, the blood pressure at the end of the preceding diastolic cycle was taken as the initial condition.

Numerical Validation

To validate our analytical solution and the Windkessel's model robustness in simulating blood pressure during a cardiac cycle, we simulated a blood flow current and plotted, using MATLAB, the results for the analytical and computational solutions for blood pressure, $P(t)$.

Figure 8 shows the aortic blood flow as simulated ($I(t)$), for 5 cardiac cycles, the analytical solution for the blood pressure $P(t)$, and the numerical solution, using an ODE solver in MATLAB (code in Appendix).
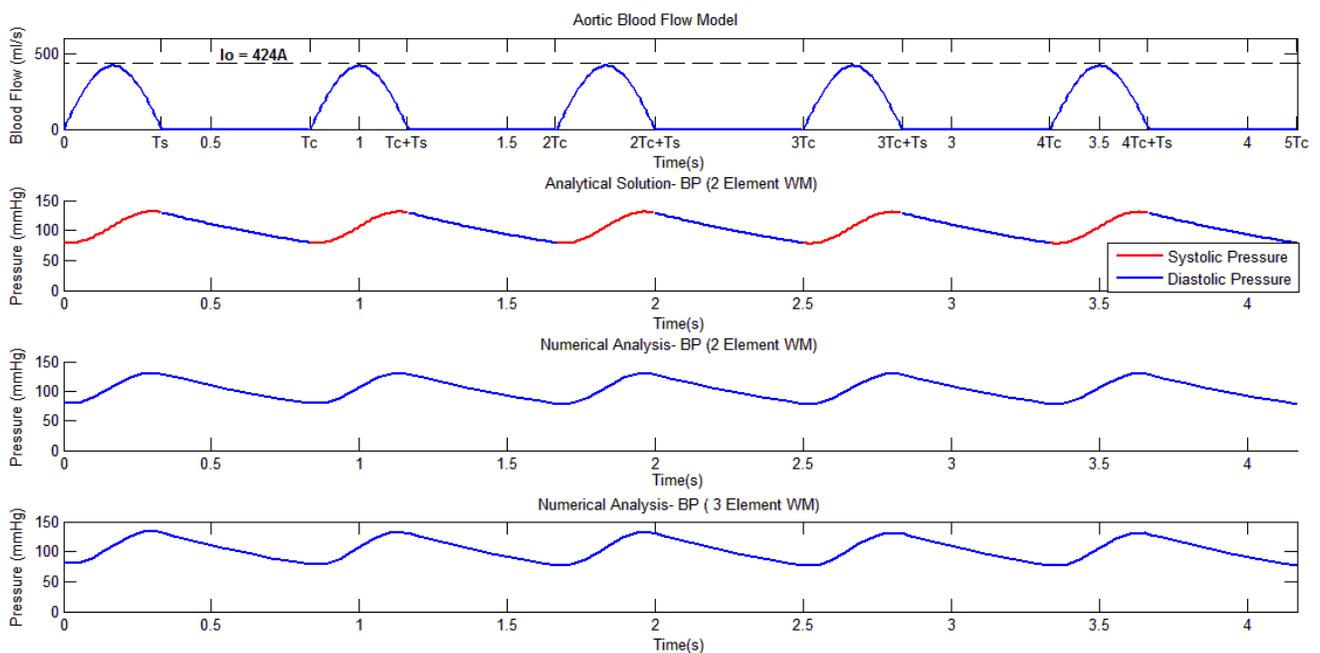

Figure 8: Analytical and Numerical solutions for 2-Element Windkessel Model.

Figure 8 shows the aortic blood pressure for the numerical solution. Note that, as expected, the blood pressure varies between the range of 80-120mmHg. Figure 9 and Figure 10 shows the numerical solution for different initial conditions, for the 2-Element Windkessel and 3-Element Windkessel Models, respectively. As time progresses, the pressure values reach equilibrium point and converge to form a single curve. Relating this physiologically, we can infer whatever the perturbation the heart is subjected to (which reflect as high or low blood pressure), it reaches steady state value after a period of time.
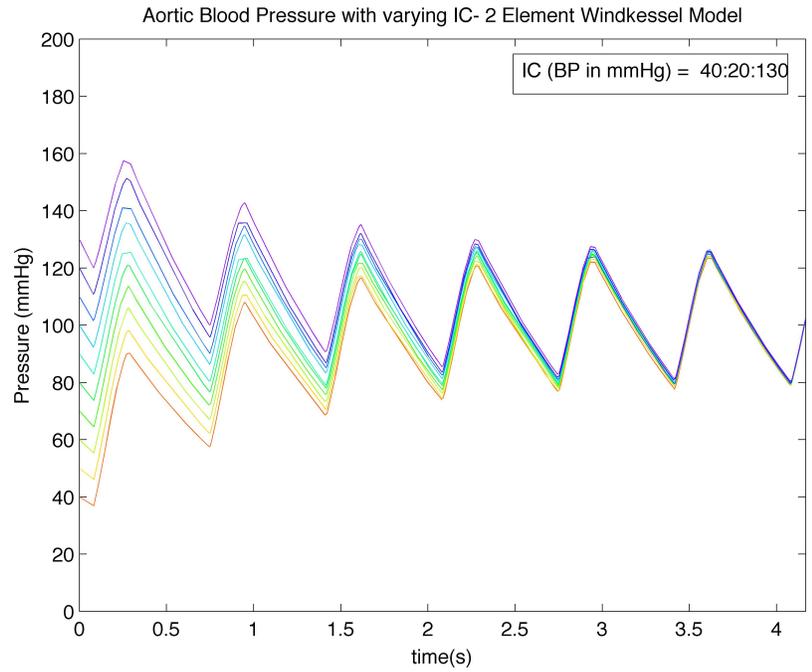
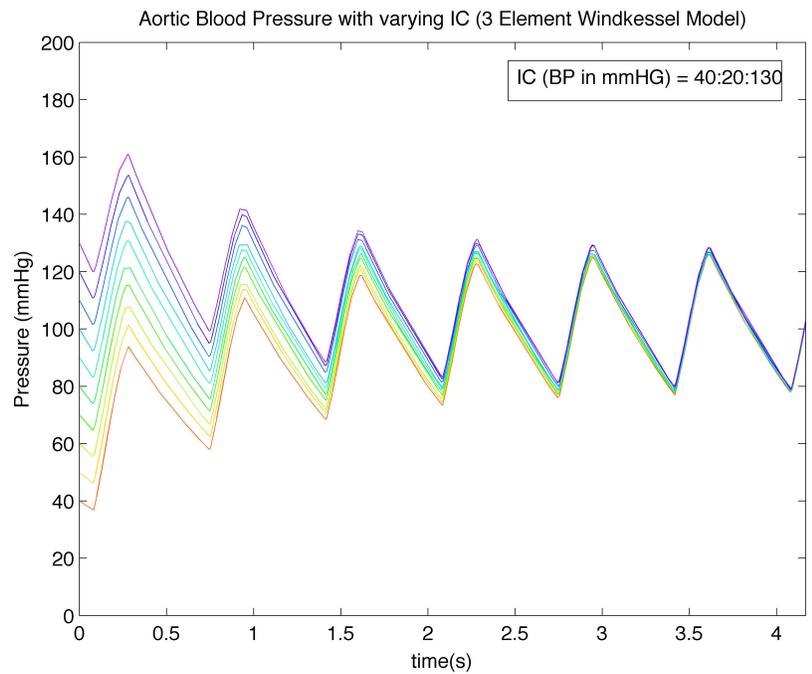Figure 9: 2-Element Windkessel Model with varying initial conditions.

Figure 10: 3-Element Windkessel Model with varying initial conditions.

Conclusion

We were able to model a healthy heart, where the blood pressure is expected to vary between 80mmHg/120mmHg during the cardiac cycle. Also, the model is capable of absorbing the fluctuations in the blood dynamics during the cardiac cycle, as seen in Figure 9 and Figure 10. Specifically, for the 2-Element WM (Figure 11), it can be observed that our analytical solution matches that of the numerical solution computed using *odesolver* in MATLAB. The results for the 2- and 3-Element match published results. However, we cannot further validate the model without patient data. Intuitively, the value of the resistance to flow (modeled in the 3-Element WM as r) is negligible when compared to the peripheral resistance (R) and hence we observe similar cardiac output for our test data. Again, our blood flow is modeled as a perfect sine, which is not an ideal representation of the aortic blood flow in real patients. However, in a real-world scenario, the three Windkessel models would represent the blood pressure better.
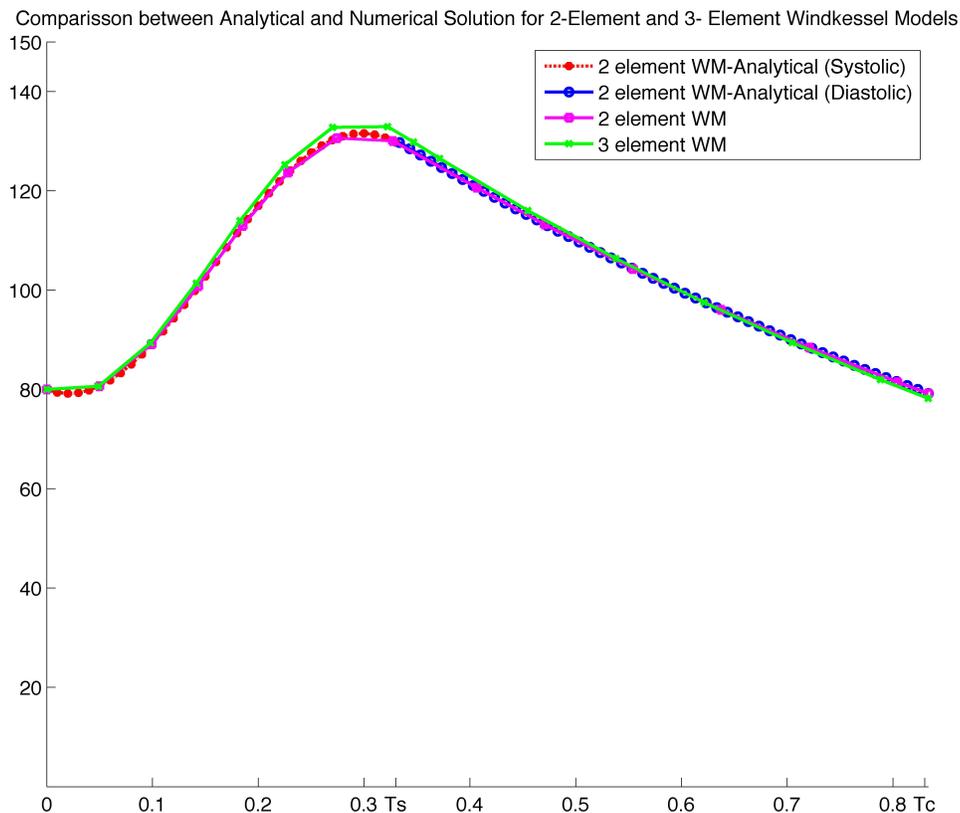

Figure 11: Comparisson of 2-Element and 3-Element Windkessel Models for Analytical and Numerical solutions.

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Appendix: MATLAB code

```matlab
close all;
clear all;
Colour=hsv;

% Defining modeling parameters for Windkessel Model

% parameters for 2 element
R = .95000 ; % systemic peripheral resistance in (mmHg/cm^3/sec)
C = 1.0666 ; % systemic arterial compliance in (cm^3/mmHg)

% parameters for 3 element
R1= .05;
R2 = .9000 ; % systemic peripheral resistance in (mmHg/cm^3/sec)
C = 1.0666;

%%Asumpltions
Tc= 60/72 ;% 72 beats per second
Ts=(2/5)*Tc; % systole period
cycle=5; % number of cardiac cycles for which WM is analysed

% Modelling blood flow to the aorta
syms ti q
I0= solve(90-int(q * (sin (pi*ti/Ts)), ti,0, Ts), q);
I0=subs(I0,'3.14',pi);
sine = @(t) sin(pi*t/Ts);
I = @(t) I0*sine(t).*(t <=  Ts); % for one cycle


figure(1)
% Analysis over 'cycle' number of cardiac cycles

for n=1:cycle
t=(n-1)*Tc:.01:n*Tc;
% Blood flow for each cardiac cycle
I = @(t) I0*sine(t-(n-1)*Tc).*(t <= ((n-1)*Tc+Ts)) ;
subplot (4,1,1)
plot (t, I(t),'LineWidth', 2)
hold on
xlim([0 n*Tc])
ylim([ 0 600])
title('Aortic Blood Flow Model')
ylabel('Blood Flow (ml/s)')
xlabel('time(s)')

%Initial conditions all models
if (n==1)
    P_ss= 80;
    P_ss2=80;
    P_ss3=80;
end


%%Analytical solution

% Analytical solution for sistolic cycle
ts=(n-1)*Tc:.01:(n-1)*Tc+Ts;
P0= P_ss + I0*Ts*R*(C*pi*R)/((Ts^2+C^2*pi^2*R^2));
P_s= @(t) P0*exp(-((t-ts(1))/(R*C))) - I0*Ts*R*(C*pi*R*cos(pi*(t-ts(1))/Ts)-Ts*sin(pi*(t-ts(1))/↩
    Ts))/(Ts^2+C^2*pi^2*R^2);
P_sd=P_s(ts(end)); % IC for the diastolic phase
```

```matlab
% Analytical  solution for diastolic cycle
td=(n-1)*Tc+Ts:.01:n*Tc;
P_d= @ (t) P_sd*exp(-(t-td(1))/(R*C));
P_ss=P_d(td(end)); % IC for the systolic phase


subplot(4,1,2)
hold on
plot(ts, P_s(ts),'r','LineWidth', 2)
plot(td, P_d(td),'b','LineWidth', 2)
xlim([0 n*Tc])
ylim([ 0 150])
title('Blood Pressure (Analytical Solution- 2 Element WM)')
ylabel('Pressure (mmHg)')
xlabel('time(s)')
legend('Systolic Pressure', 'Diastolic Pressure')


%%Numerical Solution for 2 Element WM
t=(n-1)*Tc:.01:n*Tc;
I = @(t) I0*sine(t-(n-1)*Tc).*(t <= ((n-1)*Tc+Ts)) ;
Y2= @(t,y2) (-y2/(R*C)+I(t)/C);
[t_m2,P_m2] = ode23(Y2, [(n-1)*Tc; n*Tc], P_ss2);
P_ss2=P_m2(end);
subplot(4,1,3)
hold on
plot(t_m2,P_m2,'LineWidth', 2)
ylim([ 0 150])
xlim([0 cycle*Tc])
title('Aortic Blood Pressure (Numerical Analysis- 2Element WM)')
ylabel('Pressure (mmHg)')
xlabel('time(s)')

%%Numerical Solution for 2 Element WM
Y3= @(t,y3) (-y3/(R2*C)+I(t)*(R2+R1)/(R2*C)+R1);
[t_m3,P_m3] = ode23(Y3, [(n-1)*Tc; n*Tc], P_ss3);
P_ss3=P_m3(end);
subplot(4,1,4)
plot(t_m3,P_m3,'LineWidth', 2)
hold on
ylim([ 0 150])
xlim([0 n*Tc])
title('Aortic Blood Pressure (Numerical 3 Element WM)')
ylabel('Pressure (mmHg)')
xlabel('time(s)')

%%Extracting the Blood pressure values for all model for one cycle
if (n==1)
    t2s=ts;
    P2s=P_s(ts); % Analytical solution for Systole
    t2d=td;
    P2d=P_d(td); % Analytical solution for Diastole
    t2=t_m2;
    P2=P_m2;      % Numerical solution for 2 element WM
    t3=t_m3;
    P3=P_m3;      % Numerical solution for 3 element WM
end


end


%%Analysis of Model with Various Inition Condistions
% Modelling Blood flow into Aorta
```

```
squared = @(t) (square(pi*(t-((Tc/2) - Ts))/Ts,100*(2*Ts - (Tc/2))/Tc) + 1)/2;
I_estimate = @(t) I0* squared(t).*sine(t);   %% Estimated perodic Blood flow model

% Effect of varying IC for 2 element WM
figure(3)
Y= @(t,y) (-y/(R*C)+I_estimate(t)/C);
for i= 1:10
[t_m,P_m] = ode23(Y, [0; cycle*Tc], 30+i*10);
plot(t_m,P_m,'Color',Colour(i*5,:))
hold on
ylim([ 0 200])
xlim([0 cycle*Tc])
title('Aortic Blood Pressure with varying IC- 2 Element Windkessel')
ylabel('Pressure (mmHg)')
xlabel('time(s)')
end

% Effect of varying IC for 2 element WM
figure(5)
Y= @(t,y) (-y/(R2*C)+I_estimate(t)*(R2+R1)/(R2*C)+R1);
for i= 1:10
[t_m,P_m] = ode23(Y, [0; cycle*Tc], 30+i*10);
plot(t_m,P_m,'Color',Colour(i*5,:))
hold on
ylim([ 0 200])
xlim([0 cycle*Tc])
title('Aortic Blood Pressure with varying IC (3 element)')
ylabel('Pressure (mmHg)')
xlabel('time(s)')
end

%% Comparison of WM and analytical Solutions
figure(4)
hold on
plot(t2s,P2s,'r-.*',t2d,P2d,'b-o',t2, P2, 'm-s', t3, P3,'g-x','LineWidth', 2,'MarkerSize',5);
legend('2 element WM-Analytical (Systolic)','2 element WM-Analytical (Diastolic)','2 element WM',↵
    '3 element WM')
ylim([ 0 150])
xlim([0 Tc])
```