

Mathematical Evaluation of Body Cooling Using the Heat Equation

BENG221
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Introduction

Thermoregulation is one of the body's many homeostatic processes. It is necessary because most proteins function only in a narrow temperature range. If thermoregulation fails, critical body functions can shut down as proteins denature. Primary methods of thermoregulation include sweating and vasodilation to decrease core body temperature, and vasoconstriction and shivering to increase temperature. In general this scheme of heating and cooling is effective at maintaining a core body temperature of 37°C. Additionally, the human body has adapted to many different environments be it through cultural adaptations or physiological changes.

In extreme conditions, sweating is the most effective methods of heat extraction since the evaporation of water requires a large amount of energy, but vasodilation still remains an effective method for cooling as long as external temperature is less than core body temperature. Dense beds of specialized vessels called arteriovenous anastomoses exist in different locations among species which facilitate heat transfer. Arteriovenous anastomoses bypass capillaries which shunt blood flow, allowing for large volumes of blood to pass through the surfaces of the body containing them. These regions may be thought of as “radiators” of heat[3]. In humans, these beds primarily lie in the hands, feet, and face. Vasodilation and vasoconstriction control the flow of blood to the arteriovenous anastomoses, adjusting both the volume of blood exposed to external temperatures, and the surface area over which heat exchange can occur.

One way to aid the process of changing body temperature is by changing the local temperature in regions where blood flow is significant, such as the radiator surfaces. Hospitals often achieve this by applying cold compresses to the pulse points on the body, such as the neck, elbows, and wrist[4]. Another method that has been gaining momentum is using heating or cooling ‘gloves’ that not only change the temperature near the hand, but also apply a slight vacuum to decrease pressure on the tissue and force local vasodilation. A company called AVAcore uses this vacuum supplemented perfusion to help return athletes’ core body temperature and heart rate to basal levels more quickly after exercise, allowing for extended periods of physical exertion and optimal performance in extreme conditions and reduced fatigue[5]. The technology has also been tested on military personnel and industrial workers with similar results[6].

Problem Statement

Heat transferred from our skin into the atmosphere is released through radiation, a rather ineffective and slow method of transport. Typically, cold objects are applied directly to the skin so convection, a much faster process, can transfer heat away from the body. AVAcore claims that the application of cold objects to the skin do not sufficiently facilitate heat transfer since cold temperatures cause vasoconstriction, decreasing the radiative abilities of arteriovenous anastomoses. The company also claims conventional methods are ineffective because the treatments have difficulty penetrating the body's insulating layers of tissue to facilitate heat transfer. In our model, we examine the claim that AVAcore’s device with the vacuum increased perfusion is necessary to facilitate sufficient heat transfer and subsequent cooling, or whether conventional methods can bring core body temperature down to normal from an elevated temperature in a reasonable and comparable amount of time.

In this report we model heat flow in arm under various body temperature conditions to simulate cases of exercise and rest. We perform these studies in the context of patients holding their hands against a cool object such as a peltier plate, a metal plate being cooled by water bath, or simply cold water at a constant, controllable temperature. We obtain temperature profiles over the length of the arm by solving the heat equation. Finally, we calculate total heat dissipated under various situations to quantitatively compare between our simulation conditions.

Assumptions and their Consequences

Our assumptions are as follows (Refer to figure 1 in next section):

- 1) The arm is a single, homogeneous tissue. However, some properties bone, muscle, fat, blood, and skin are all incorporated into the model partially via the constants used for simulation. Furthermore, all components of the homogeneous tissue are static. Hence we do not account for blood flow in our model.
- 2) We assume no radial heat transfer. Rather, heat may only travel down the axis of the arm from body to hand, which we denote as the x -axis. Consequently, our problem is a 1D heat transfer problem.
- 3) The only heat exchange of the arm with its “surroundings” are with the rest of the body, which we denote $x=0$, and with the cooling object at the hand, which we denote $x=L$. Furthermore, the exchange is instantaneous, allowing for value boundary conditions at both $x=0$ and $x=L$. In particular, the temperature at the beginning of the arm ($x=0$) is equal to the core body temperature. Similarly, the temperature at the hand ($x=L$) is equal to the temperature of the cooling object.
- 4) As previously stated, we consider a state of constant exercise. This, along with assumption 3, means that the core body temperature does not fall, despite cooling or heat dissipation from the hand. Rather, it stays at an elevated body temperature throughout the time we consider.
- 5) We disregard the possibility of metabolic heat production, assuming that the change in the temperature distribution of the arm from the initial condition is caused by the temperature gradient from the body to the hand.

Mathematical Expression and Analytical Solution

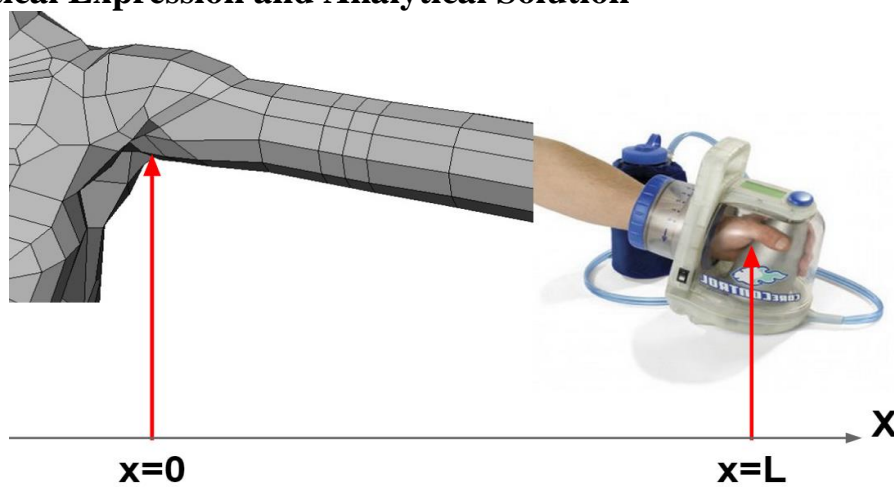


Fig. 1: Spatial definitions of coordinates along the length of the arm

Governing Equation

With our assumptions in mind, we mathematically model our problem using the heat equation. The most general statement of this problem in cylindrical coordinates is

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(k r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + Q$$

By assumption 2, the first and second terms of the right hand side of the equation are neglected. That is, we say that the temperature is not dependent on r or ϕ , and so their derivatives become zero.

$$\rho c \frac{\partial T}{\partial t} = \cancel{\frac{1}{r} \frac{\partial}{\partial r} \left(k r \frac{\partial T}{\partial r} \right)} + \cancel{\frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right)} + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + Q$$

Similarly, the generation term, Q , becomes 0 by assumption 5. Note in the following equation that we have changed the variable z to x for convenience, as our problem has now become a 1-dimensional problem.

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} + \cancel{Q}$$

Note also in the equation above that we have divided both sides of the equation by ρc , which gives the constant

$$D = \frac{k}{c\rho}$$

where c is the heat capacity of the tissue, k is the thermal conductivity of the tissue, and ρ is the tissue's mass density. D is the thermal diffusivity.

At last we are left with the 1D heat equation with no generation term.

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

This heat equation was analyzed to determine the temperature distribution over the length of the arm over a given time.

Boundary and Initial Conditions

In accordance with figure 1 and our problem conditions stated in the assumptions, the boundary and initial conditions were defined as follows:

$$T(0, t) = T_o$$

$$T(L, t) = T_L$$

$$T(x, 0) = T_b$$

where the temperature at start of the arm is assumed to be equivalent to elevated core body temperature T_o ; the end of the arm at the device hand is assumed to be the temperature of the cooling plate, T_L ; and the arm is assumed to initially be some temperature T_b for all x . These follow from assumptions 3 and 4.

Analytical Solution

Addressing Inhomogeneity

The boundary conditions are nonhomogeneous. This precludes direct use of separation of variables to solve the PDE. Instead, the general solution will be the sum of a "homogeneous" and "particular" solution as follows:

$$T(x, t) = v_H(x, t) + T_p(x)$$

where v_h is the homogeneous solution and T_p is the particular solution.

Particular Equilibrium Solution

The particular solution accounts for the boundary condition inhomogeneity. It is akin to finding a steady state or equilibrium temperature distribution. Hence $\partial T/\partial t=0$, and the equation reduces simply to the following equation

$$\frac{dT_p^2}{dx^2} = 0$$

The boundary conditions remain as stated for the overall problem. We ignore the initial condition in the steady state calculation. This is a simple second order ODE. Integrating twice yields the linear solution

$$T_p = Ax + B$$

Applying the boundary conditions, the coefficients A and B can be determined as follows.

$$\begin{aligned} T_p(0, t) &= T_o = A(0) + B \\ T_p(L, t) &= T_L = A(L) + B \\ B &= T_o \\ A &= \frac{T_l - T_o}{L} \end{aligned}$$

The final equilibrium temperature distribution is

$$T_p(x) = \frac{T_l - T_o}{L}x + T_o$$

Homogenous Problem - Displacement from Equilibrium and Separation of Variables

We now solve the homogeneous problem for $v(x,t)$. $v(x,t)$ is the temperature displacement from the equilibrium temperature.

$$v_H(x, t) = T(x, t) - T_p(x)$$

The boundary conditions must be expressed for the displacement $v(x,t)$. They may be found by substitution.

$$\begin{aligned} v(0, t) &= T_o - \frac{T_L - T_o}{L}(0) - T_o = 0 \\ v(L, t) &= T_L - \frac{T_L - T_o}{L}L - T_o = 0 \end{aligned}$$

For the displacement $v(x,t)$, the boundary conditions are homogeneous and linear. In order to use separation of variables, the equation itself must also be homogeneous and linear. Because $\partial T/\partial t = \partial v/\partial t$ and $\partial^2 T/\partial x^2 = \partial^2 v/\partial x^2$ (since the equilibrium solution is only a function of x), $v(x,t)$ must also satisfy the heat equation as previously stated.

$$\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2}$$

This is a linear homogeneous PDE. Therefore we can use separation of variables to solve it. We assume that the form of the solution is of the product form

$$T(x, t) = G(t)\Phi(x)$$

Taking the first and second derivatives of $v(x,t)$ with respect to time and space respectively, substituting them into the PDE, and finally dividing both sides by $G(t)$ and $\Phi(x)$ separates the variables and shown below.

$$\phi \frac{dG(t)}{dt} = DG \frac{d^2\phi(x)}{dx^2} = -\lambda$$

$$\frac{1}{DG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda$$

Note that we also rearranged the constants for convenience. Because G is a function only of t and Φ is only a function of x, the only way the derivative expressions above may be equal is if both expressions equal some constant. We select $-\lambda$ for convenience in finding the eigenvalues.

Spatial Component

The spatial component is shown to be a second order ODE:

$$\phi'' = -\lambda\phi$$

$$\phi'' + \lambda\phi = 0$$

To find nontrivial solutions, cases must be considered based on the value of λ , the eigenvalue.

Case 1

$$\lambda = 0$$

$$\phi'' + \lambda\phi = 0$$

$$\phi'' = 0$$

$$\phi = Ax + B$$

boundary conditions are applied, and a trivial solution is obtained.

$$\phi(0) = 0 = A(0) + B$$

$$B = 0$$

$$\phi(L) = 0 = A(L) + B$$

$$A = 0$$

Case 2

$$\lambda < 0$$

$$\phi'' + \lambda\phi = 0$$

$$\phi'' - |\lambda|\phi = 0$$

$$r^2 - |\lambda| = 0$$

$$r = \pm\sqrt{|\lambda|}$$

By applying boundary conditions, another trivial solution is obtained.

$$\phi(0) = 0 = C_1 + C_2$$

$$C_1 = -C_2$$

$$\phi(0) = 0 = C_1 e^{L\sqrt{|\lambda|}} + C_2 e^{-L\sqrt{|\lambda|}}$$

$$0 = C_1 (e^{L\sqrt{|\lambda|}} - C_2 e^{-L\sqrt{|\lambda|}})$$

$$C_1 = C_2 = 0$$

Case 3

$$\begin{aligned}\lambda &> 0 \\ \phi'' + \lambda\phi &= 0 \\ \phi'' + |\lambda|\phi &= 0 \\ r^2 + |\lambda| &= 0 \\ r &= \pm i\sqrt{\lambda}\end{aligned}$$

The solution, along with the Euler expansion, is as follows:

$$\phi = K_1 e^{i\sqrt{\lambda}x} + K_2 e^{-i\sqrt{\lambda}x}$$

$$\phi = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

Boundary conditions are applied:

$$\begin{aligned}\phi(0) = 0 &= C_1 \cos(0\sqrt{\lambda}) + C_2 \sin(0\sqrt{\lambda}) \\ 0 &= C_1 \cos(0) \\ C_1 &= 0\end{aligned}$$

$$\begin{aligned}\phi(L) = 0 &= C_1 \cos(L\sqrt{\lambda}) + C_2 \sin(L\sqrt{\lambda}) \\ 0 &= C_2 \sin(\sqrt{\lambda}L)\end{aligned}$$

In order to satisfy the relationship while maintaining non-triviality, C_2 cannot equal zero; therefore, λ must be some value which allows the sine expression to equal zero.

$$\begin{aligned}\sqrt{\lambda}L &= \pi n \text{ for } n = 1, 2, 3, \dots \\ \lambda &= \left(\frac{\pi n}{L}\right)^2\end{aligned}$$

The corresponding spatial component of the solution is then

$$\phi(x) = A_n \sin\left(\frac{n\pi}{L}x\right) \text{ for } n = 1, 2, 3, \dots$$

Temporal Component

Analysis of temporal component by separation of variables shows that the solution $G(t)$ is an exponential expression

$$\begin{aligned}\frac{1}{G} \frac{dG}{dt} &= -\lambda \\ \frac{dG}{G} &= -D\lambda dt \\ G &= B e^{-D\lambda t}\end{aligned}$$

Determining Constant Term

Multiplying the temporal and spatial components provides the general solution for the displacement $v(x,t)$:

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 D t}$$

Now the initial condition is used to determine constant A_n . As with the boundary conditions, we must determine the initial condition for $v(x,t)$ from the initial condition for $T(x,t)$. This new

initial condition is simply the the initial condition of $T(x,t)$ minus the the steady state temperature solution. With this knowledge, we now substitute $t=0$ and obtain the following expression for solving the constant A_n .

$$f(x) - u_p = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

Next we multiply both sides by $\sin(n\pi x/L)$ and integrate both sides of the equation over the domain of the problem $x = [0,L]$.

$$\int_0^L [f(x_o) - u_p] \sin\left(\frac{n\pi}{L}x_o\right) dx_o = \sum_{n=1}^{\infty} A_n \int_0^L \sin^2\left(\frac{n\pi}{L}x_o\right) dx_o = \frac{A_m L}{2}$$

In the above expression we applied the orthogonality of the sine function to obtain the right most expression. Proceeding, we solve for the constant by substituting the initial condition for $T(x,t)$ and the equilibrium temperature distribution.

$$A_m = \frac{2}{L} \int_0^L [f(x) - u_p] \sin\left(\frac{n\pi}{L}x_o\right) dx_o$$

$$A_m = \frac{2}{L} \int_0^L \left(T_b - \frac{T_L - T_o}{L}x_o - T_o\right) \sin\left(\frac{n\pi}{L}x_o\right) dx_o \text{ where } R_1 = -\frac{T_L - T_o}{L} \text{ and } R_2 = T_b - T_o$$

$$A_m = \frac{2}{L} \int_0^L (R_1 x_o + R_2) \sin\left(\frac{n\pi}{L}x_o\right) dx_o$$

To solve this integral, we must integrate by parts.

$$\begin{aligned} u &= R_1 x + R_2 & v &= -\left(\frac{L}{n\pi}\right) \cos\left(\frac{n\pi}{L}x\right) \\ du &= R_1 dx & dv &= \sin\left(\frac{n\pi}{L}x_o\right) dx_o \end{aligned}$$

$$\begin{aligned} A_m &= \frac{2}{L} \left[uv - \int_0^L v du \right] = \frac{2}{L} \left[(R_1 x + R_2) \left(-\frac{L}{n\pi}\right) \cos\left(\frac{n\pi}{L}x\right) \Big|_0^L + \int_0^L \left(\frac{L}{n\pi}\right) \cos\left(\frac{n\pi}{L}x_o\right) R_1 dx_o \right] \\ &= \frac{2}{L} \left[(R_1 x + R_2) \left(-\frac{L}{n\pi}\right) \cos\left(\frac{n\pi}{L}x\right) \Big|_0^L + R_1 \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{L}x\right) \Big|_0^L \right] \\ &= \frac{2}{L} \left[-(-T_L + T_b) \left(\frac{L}{n\pi}\right) \cos(n\pi) + (T_b - T_o) \left(\frac{L}{n\pi}\right) \right] \end{aligned}$$

Finally, the final analytical solution for the original heat equation is

$$T(x, t) = T_p + v_H = \frac{T_L - T_o}{L}x + T_o + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$

where A_n is given by the above expression A_m .

Temperature Profiles - Analytical and Numerical Solutions

Constant values that define D must be determined in order to plot the analytical solution. The values used are presented in **Table 1** and are based on previously established values from literature.

C , Specific Heat Capacity	$2.91 \frac{\text{J}}{\text{g}^\circ\text{C}}$
k_o , Thermal Conductivity	$0.013 \frac{\text{W}}{\text{cm}^\circ\text{C}}$
ρ , Mass Density,	$1.059 \frac{\text{g}}{\text{cm}^3}$
L , Length of the Arm	75 cm
r , Radius of the Arm	5 cm
T_b , Initial Condition along the arm	= T _o
T_o , Elevated Body Temperature Simulate Exercise	varies
T_L , Hand/Device Temperature	varies

Table 1. Table of Constants

Specific heat capacity[7], thermal conductivity[8], and mass density[9] are estimated based on a combination of published values for different tissue types. In particular, we found specific heats, thermal conductivities, and mass densities for different components of the arm, including bone, fat, skin, blood, and muscle, and took average values of the individual quantities as our constants for simulation. Our boundary conditions and initial condition vary for different scenarios we present below, so we refrain from setting specific values here. However, we explain now a few important temperatures that we used and explain their significance. The boundary condition of core body temperature equal to 39°C represents an extreme physiological limit. A person who is feverish should not reach this temperature. However, this number would not be unusual for an individual under great physical exertion, such as a soldier in a desert environment, a firefighter at the scene of the fire, or an athlete in play. The hand temperature of 15°C could be achieved when the hand is in contact with a cooling object. The initial condition, T_b, is selected to be equal to the core body temperature (T_b=T_o).

Analytical Solutions

Plots are made with different numbers of terms (2, 10, and 1000) from the analytical solution using the same boundary and initial conditions. The MATLAB code used to generate these are in Appendix A. The boundary and initial conditions for the solutions shown in figures 2-4 are T_b=T_o=39°C and T_L=15°C. Analytical solutions were also generated for the boundary and initial conditions to be described in figures 5-6 and 8, but they are not shown. They are discussed briefly in the discussion section. For the axes in all figures 2-14, the units are seconds for time, cm for x, and °C for T(x,t) (denoted u(x,t) in the plots).

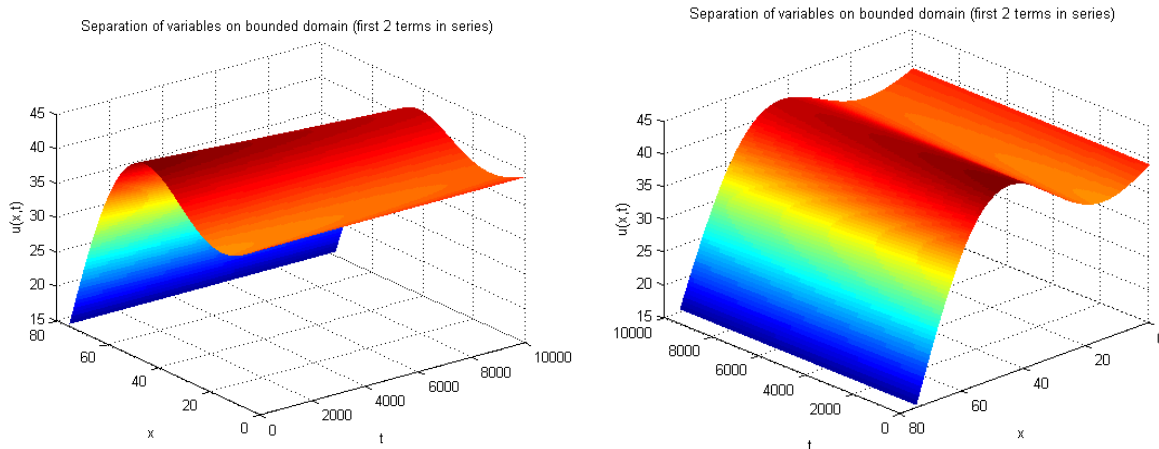


Fig. 2. Surf plot of first 2 terms from two different angles

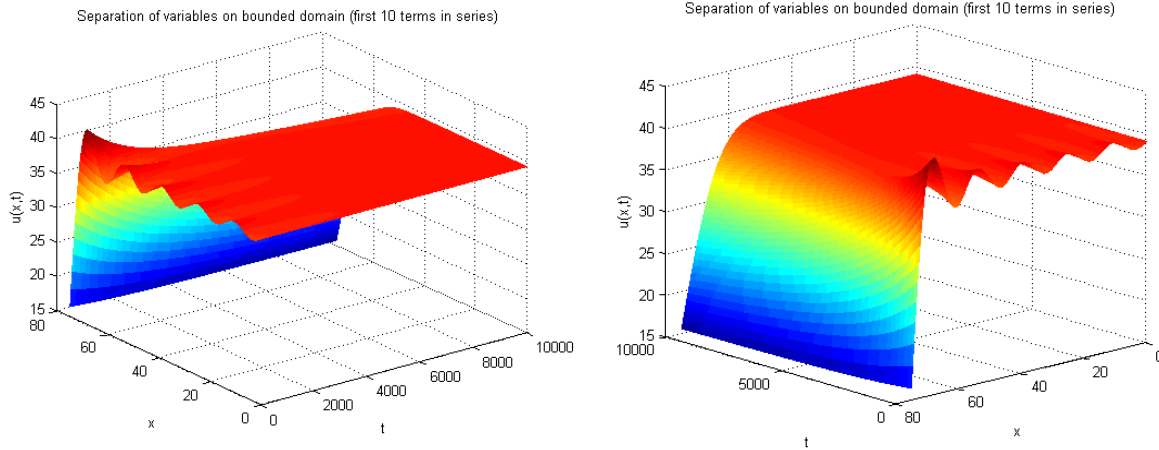


Fig. 3. Surf plot of first 10 terms from two different angles

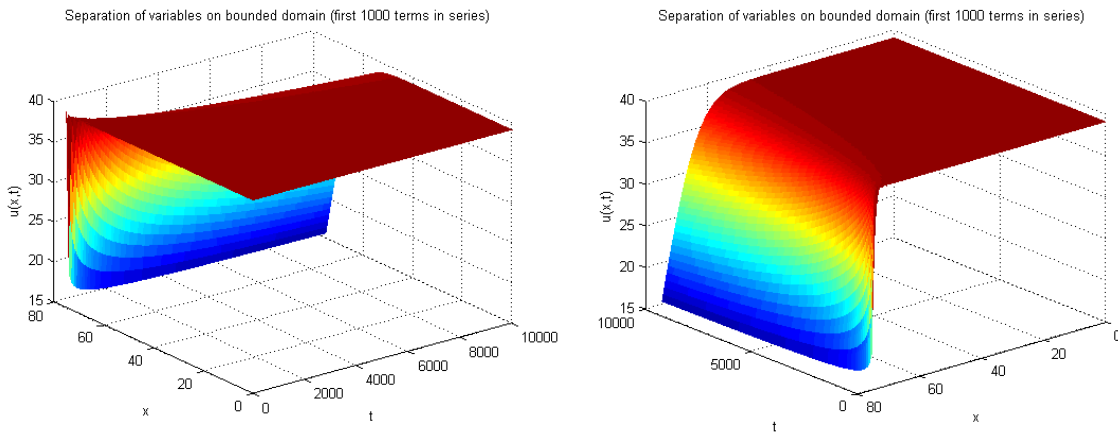


Fig. 4. Surf plot of first 1000 terms from two different angles

Numerical Solution

MATLAB's built in pdepe is used to generate the surf plots in this section (Appendix B). Different scenarios are examined by altering the problem's boundary conditions. The total heat dissipated is also analyzed in each case to further study the effectiveness of the cooling in the arm simply by holding a cold object.

Since pdepe computes the solution at every length interval of dx along the arm for a given time interval, we can use the trapezoidal rule to approximate the integral of the temperature distribution at a particular time point and hence calculate the heat dissipated.

$$\text{Heat lost} = dx_{\text{arm}} * \left(T_{\text{arm}} - \sum_{n=1}^{nt-1} \frac{(T_n + T_{n+1})}{2} \right) * c * \rho * r_{\text{arm}}^2 * \pi$$

We take the average temperature between every interval and subtract from initial temp. This is multiplied by the length interval, the heat capacity of the tissue, tissue density and the cross sectional area of the arm (Table 1). The arm radius was approximated as 5 cm to account for a bulkier athlete, who would be a likely candidate for using a cooling device

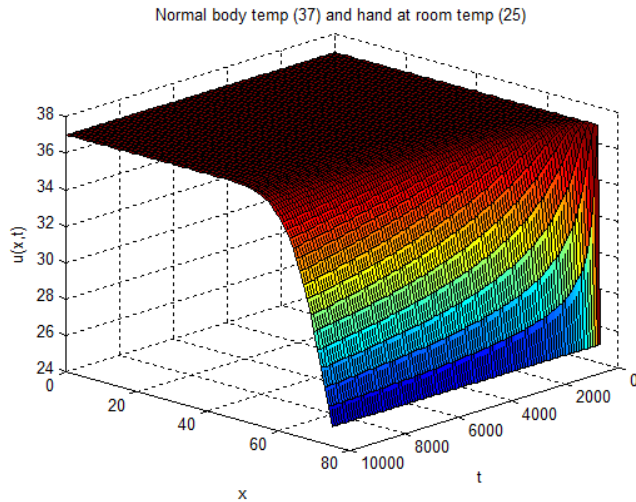


Fig. 5. This first case models the hand temperature profile at a basal condition assuming normal body temperature ($T_o=37^\circ\text{C}$) and room temperature surroundings ($T_L=25^\circ\text{C}$). A cooling device is not considered for this situation. Using our integration scheme, we determine the total heat dissipated at some final time. Here, we consistently set this value at $t=10,000\text{ s} \sim 3\text{hrs}$, to exaggerate the value. Given these conditions, $Q_{\text{lost}} = 2.2182\text{e}4\text{ J}$ at $t=10,000\text{s}$. Also, the theoretical total heat in the arm initially for the core body temperature of $T_L=T_o=37^\circ\text{C}$ is $Q_{t_0}=6.9390\text{e}5\text{ J}$. The value used for this

theoretical total changes in the following cases to match the initial condition used. This is basically the amount of heat in the arm at initial condition. It is calculated simply by multiplying the value of the initial condition by the full length of the arm L and then multiplying by the heat capacity of the tissue, tissue density and the cross sectional area of the arm.

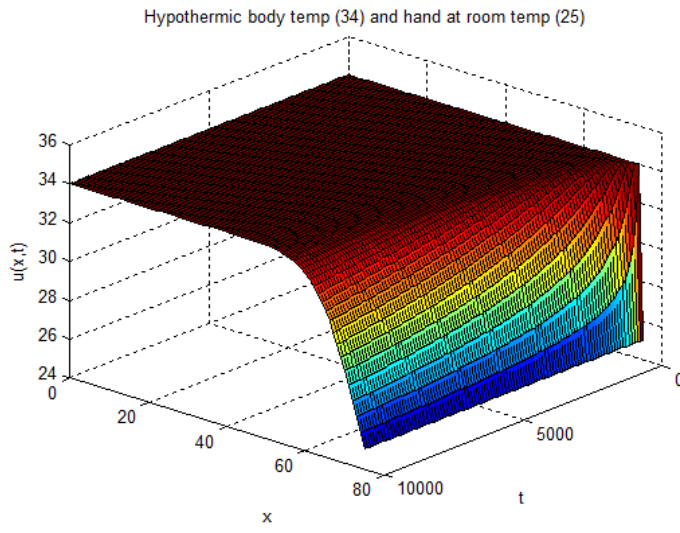


Fig. 6 (Middle). To verify the usefulness of our model, we next examined an irregular case where the body is at a near hypothermic temperature of $T_o=34^\circ\text{C}$ and examined the temperature profile. $Q_{\text{lost}}=1.6636\text{e}4\text{ J}$, which is reasonable considering the body is cooler and will dissipate less heat than when it is at the basal condition. Theoretical heat in the arm was calculated as $Q_{t_0}=6.3764\text{e}5\text{ J}$ based on the initial condition.

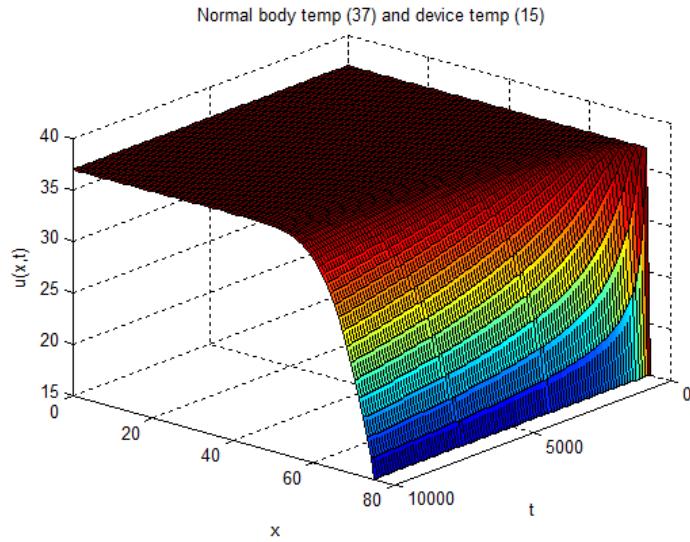


Fig. 7. Now we finally look at the relevant situation to study cooling in the arm simply by applying a cool object to the hand during physical exertion. The body temperature is elevated to $T_o=T_b=39^\circ\text{C}$. The cooling device constrains the temperature at the hand to $T_L=15^\circ\text{C}$. Using the same integration scheme as before we find $Q_{\text{lost}}=4.4364\text{e}4$ J. This is considerably more than the previous schemes. The Q_{to} is also higher, $7.3141\text{e}5$ J, since there is more heat in the body (and arm due to the initial condition).

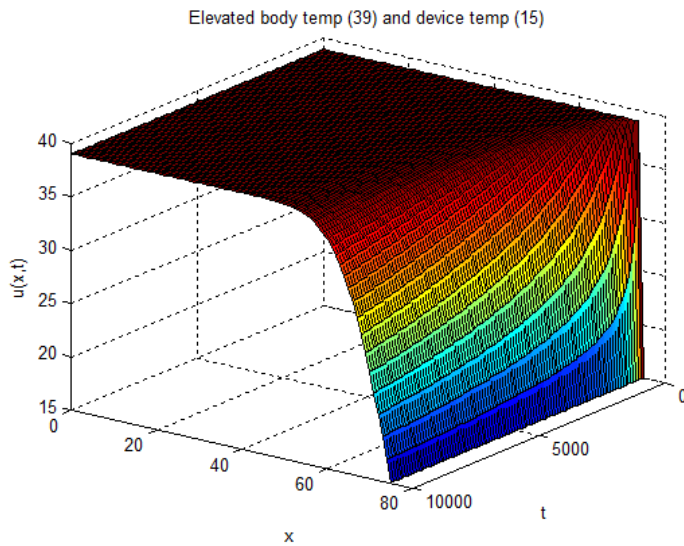


Fig. 8. Lastly, we look at the case where a person at normal body temperature uses the device. Even at a normal body temperature, we see a much greater dissipation of heat compared to the basal condition. Here, $Q_{\text{lost}}=4.0667\text{e}4$ J and $Q_{\text{to}}=6.9390\text{e}5$ J. The implications of these findings will be further discussed below.

Scenario	$T_b (= T_o)$ ($^\circ\text{C}$)	T_L ($^\circ\text{C}$)	Q_{lost} (J)	Q_{lost} (kcal)	Total Initial Heat in Arm (J)
Figure 5	37	25	2.2182e4	5.30	6.9390e5
Figure 6	34	25	1.6636e4	3.98	6.3764e5
Figure 7	39	15	4.4364e4	10.60	7.3141e5
Figure 8	37	15	4.0667e4	9.72	6.9390e5

Table 2: Summary of heat loss and boundary conditions for different simulation cases

Effect of Parameters on Analytical Solution

We also wish to observe how changes in the parameter D might affect the analytical solution. In figures 9-14, we present the analytical solution using the first 1000 terms and with the boundary conditions $T_o=T_b=39^\circ\text{C}$ and $T_L=15^\circ\text{C}$. Hence all these surface plots may be compared to Fig. 4, for which the boundary conditions were the same, $D=0.0042\text{ cm}^2/\text{s}$, $L=45\text{ cm}$, and k , c , and Φ values equivalent to those in table 1.

Figures 9-11 use values of thermal conductivity and specific heat capacity specific to various tissues in the arm.

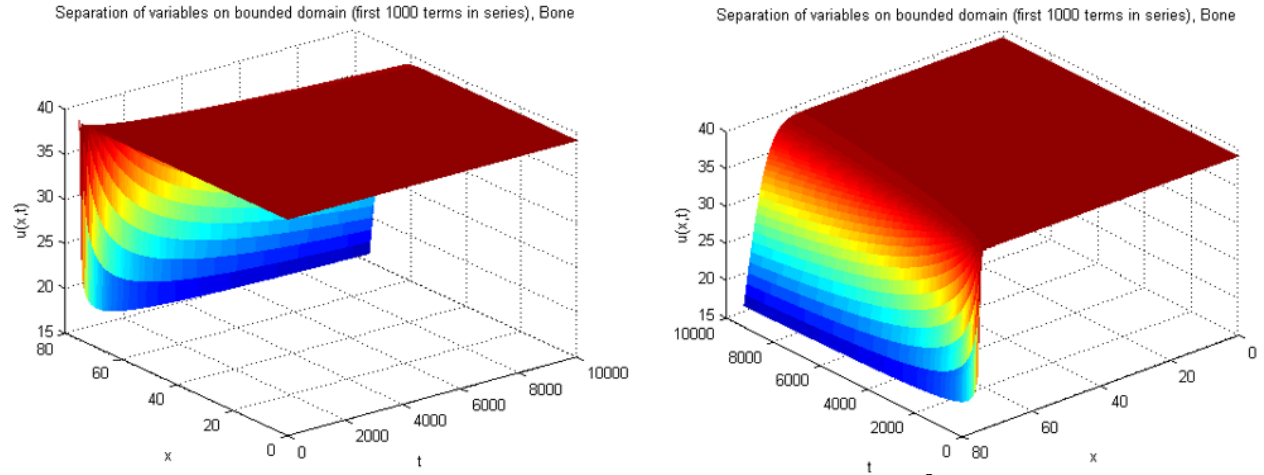


Fig. 9: Surface plot using bone properties $c=1.3\text{ J/g}^\circ\text{C}$, $\Phi=1.99\text{ g/cm}^3$, $k=0.00373\text{ W/cm}^\circ\text{C}$, $D=0.001\text{ cm}^2/\text{s}$

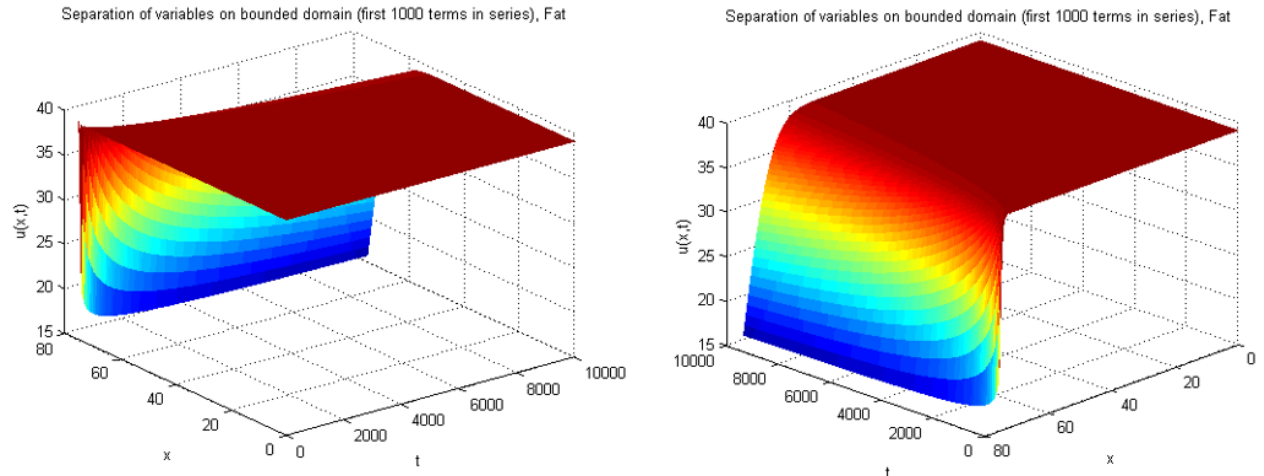


Fig. 10: Surface plot using fat properties $c=2.25\text{ J/g}^\circ\text{C}$, $\Phi=0.916\text{ g/cm}^3$, $k=0.00505\text{ W/cm}^\circ\text{C}$, $D=0.002\text{ cm}^2/\text{s}$

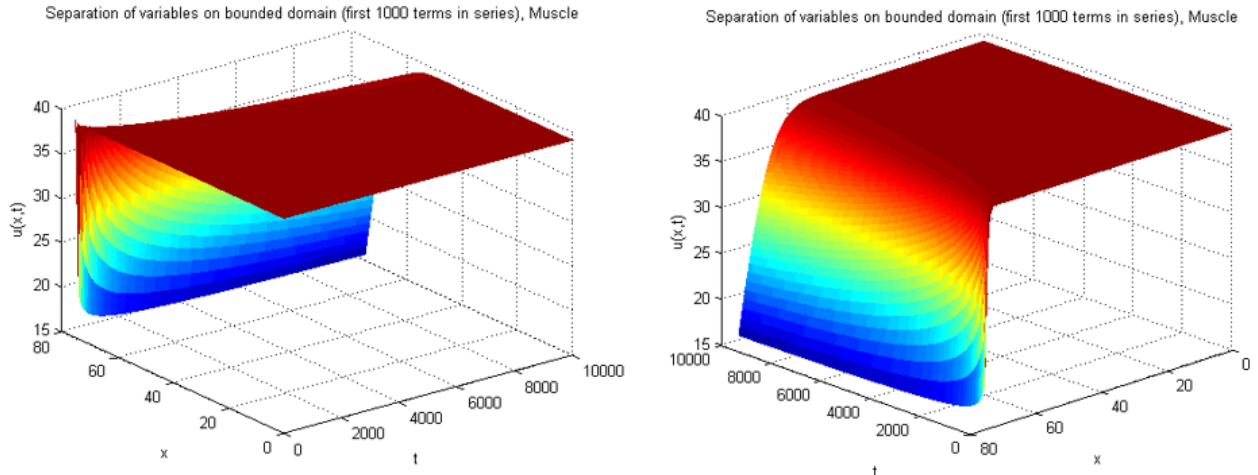


Fig. 11: Surface plot using muscle properties $c=3.72 \text{ J/g}^\circ\text{C}$, $\Phi=1.059 \text{ g/cm}^3$, $k=0.01235 \text{ W/cm}^\circ\text{C}$, $D=0.003 \text{ cm}^2/\text{s}$

Figures 12-14 show different values of k_0 which significantly change the value of the D .

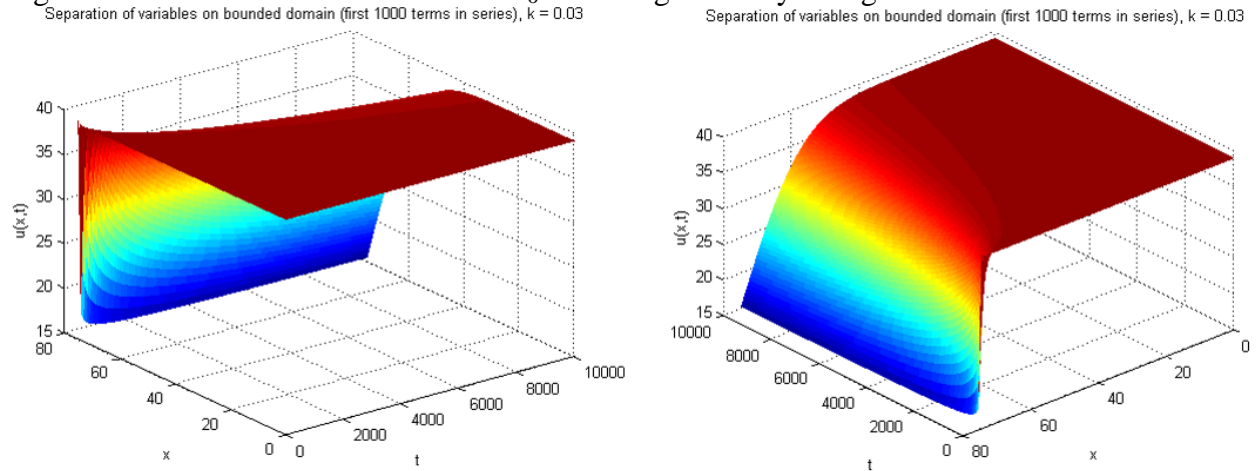


Fig. 12: Surface plot for $D=0.0097 \text{ cm}^2/\text{s}$ due to the increase in k

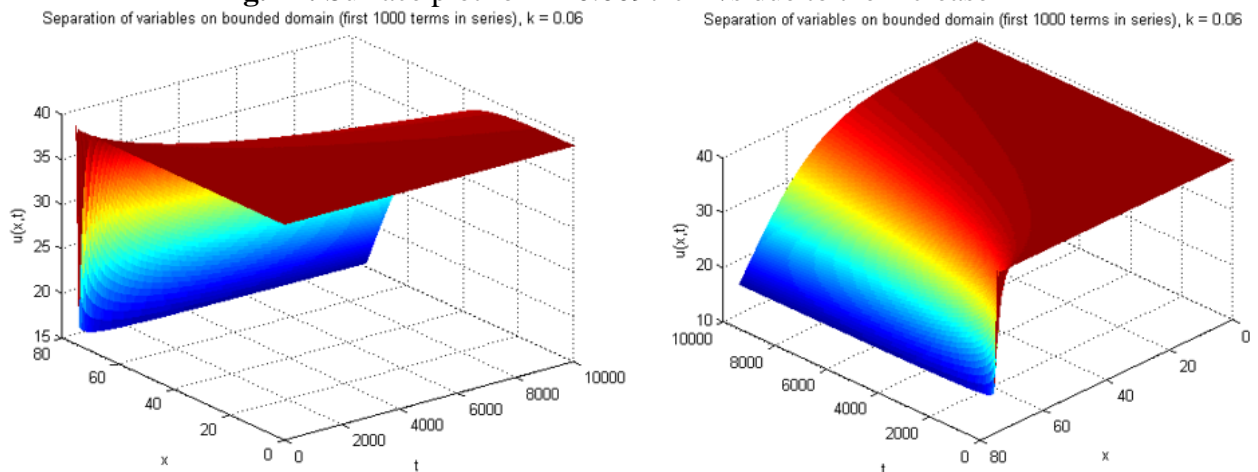


Fig. 13: Surface plot for $D=0.0195 \text{ cm}^2/\text{s}$ due to the increase in k

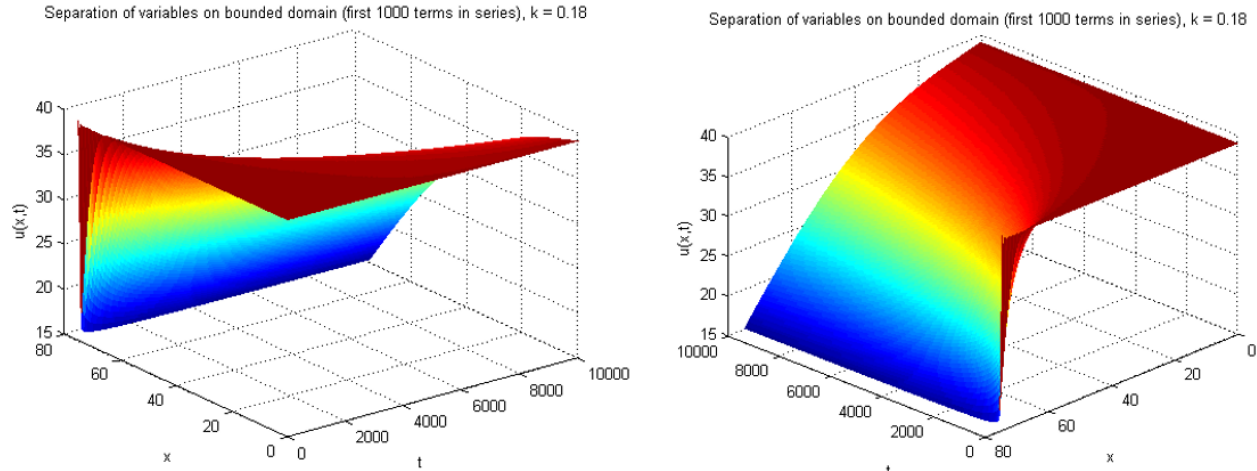


Fig. 14: Surface plot for $D=0.0584 \text{ cm}^2/\text{s}$ due to the increase in k

Discussion

The analytical graphs were compared to the numerical graphs under every set of conditions in figures 5-8. Upon qualitative examination, the surfaces produced from the analytical solution seemed to be in agreement with the numerical solutions from pdepe. At a low number of terms, the match was crude and the individual sine terms created oscillations in the surf plots (Figures 2 and 3). We increased the number of terms and found that at $n=1000$, the two plots were qualitatively identical. Another means of increasing the accuracy of our analytical solution surfaces would have been to decrease the step size of x and t in the pdepe function i.e. increase the discretization. However, this would have required significantly increased the computational cost and time.

In the heat dissipation scenarios, the heat lost was a small portion of what was in the arm at initial condition, even over long time intervals. Referring to Table 2, the heat lost from the hand at room temperature, our basal condition, was found to be around 5.3 kcal. Considering that this is just a small portion of the average person's daily intake ($\sim 2,500$ kcal), this is a reasonable number. Later, when we used our integration scheme to find the heat lost by cooling the hand to 15°C and at normal body temperature. This resulted in a value of 9.7 kcal. Thus, we can conclude that cooling the hand from a higher temperature initial and core body temperature does indeed dissipate more heat. In order to phrase this in terms of using the AVAcore device, we compute how much heat loss is required to lower core body temperature from 39° to 37°C (elevated to normal body temperature). The heat loss required is $3.7508 \times 10^4 \text{ J}$ or 8.96 kcal. Our model over 3 hours simulating the AVAcore device at 15°C and at normal body temperature estimated 10.6 kcal lost by the end of the time course. To give a more precise time to dissipate excess heat, it was determined by MATLAB that it takes 2.42 hours, an unreasonable time for an athlete who needs to be ready to compete as soon as possible. Therefore, we want to determine whether the AVAcore is a significant improvement. In the following experiments from literature, the AVAcore device was used on 6 different test subjects and had their esophageal temperature measured.

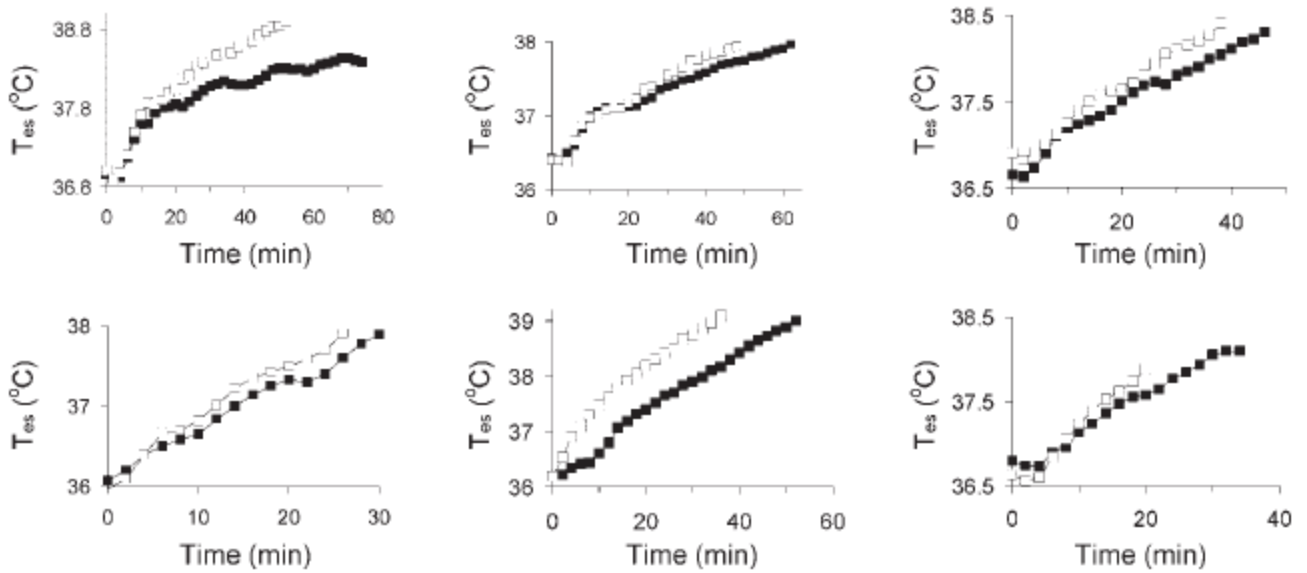


Fig. 9. Esophageal Temperature TM (° C) versus Time (min) for 6 test subjects[10]

We find that, under constant exercise, the AVAcore device reduces the subjects temperature by about 0.5-1° C over a period of ~40 minutes. Using our code, we can check the amount of time for our model to dissipate the same amount of heat to make a 1 degree difference. We found that it took 38.3 minutes to do the same as the AVAcore, suggesting that there is little difference between their device and simply using a cold object to cool the hand. In other words, the same cooling may be achievable without the vacuum effect that AVAcore claims is essential to sufficient cooling and resulting performance enhancements. However, the main difference is that our model does not account for heat generated by the body as the user is exercising.

However, there are many qualifications we have to make about our model, despite results that make physical sense. For one, the arm is not a homogenous tissue, yet we averaged heat capacity and density across tissue types to approximate our results. The cross sectional area is not a perfect circle nor is it constant, but we chose to approximate the entire arm as a cylinder. Another major assumption was that heat flowed in only one dimension to simplify our PDE model. Thus, other pathways of heat regulation were neglected. This includes homeopathic functions of the body to self-regulate heat and local tissue heat generation. Also, heat is carried predominantly by the blood convection and perfusion through the tissue, but we had no way of modeling this with the relevant knowledge from this course. The Pennes bioheat transfer model is the most well known way of modelling heat transfer in the body and would be preferred in more rigorous models. The device itself also features a vacuum feature that increases blood perfusion, which we could not account for in our model. Other things we did not take into account for the device was surface area in contact and heat flux into the device. Since we did not show significant improvement from the device, more accurate models would be desired to justify continuing practice in the athletic world.

Finally we wanted to consider how changes in our parameter D affect the analytical solution. From figures 9-14, we see increasing values of D. Figures 9-11 show the changes in D

related with different tissues. D increased from bone to fat to tissue. Considering the surface plots, there was only a very small change from one to the other. This makes sense, however, because the change in D from tissue to tissue was not very large. However, there was a slight change in that the curves more quickly approached the steady state solution when D was larger. This change, however, can be very clearly seen in figures 12-14. Though these values of D were arbitrarily chosen and do not necessarily represent physiological values of D , the surface plot approached the steady state linear solution far more quickly when D was significantly larger. From this result, we may also conclude individually that an increase in any value to which D is proportional will cause the analytical solution to approach the steady state temperature distribution more quickly. This means that heat transfer in the arm occurs more quickly, resulting in faster redistribution of energy. Conversely, increases in parameters inversely proportional to D will cause the solution to approach the steady state temperature distribution more slowly. In other words, it will slow the transfer of heat energy through the arm. Hence increases in k_o will quicken the approach to the linear, steady state temperature distribution and increases in c and Φ will slow it.

Appendices

Appendix A: Matlab Code for Analytical Solution Plots

```
clear all; close all; clc;
ns = 1000;

c = 2.91; %j/degC*g
rho = 1.059900 ; %g/cm^3
k = 0.0132; %W/degC*cm

global D
D = k/(c*rho);
global T_L
T_L = 15; %degC
global T_B
T_B = 39; %degC
global T_0
T_0 = 39; %degC
global L
L = 75; %cm

% domain
dx = 0.4; % step size in x dimension
dt = 100; % step size in t dimension
xmesh = 0:dx:L; % domain in x
tmesh = 0:dt:10000; % domain in t
nx = length(xmesh); % number of points in x dimension
nt = length(tmesh); % number of points in t dimension
%A = (T_L-T_0)/L; B = T_0;

% solution on bounded domain using separation of variables
sol_sep = zeros(nt, nx);
for n = 1:ns
    lambda = pi*n/L;
    An = -(2/L)*(-T_L+T_B)*(1/lambda)*cos(lambda*L) + (T_B -
T_0)*(1/lambda);
    %An = 1/(pi^2*n^2)*(L*(pi*n*cos(pi*n))*(-
(A*L+B))+A*L*sin(pi*n)+pi*B*n);
    sol_sep = sol_sep + An*exp(-D*(lambda^2)*tmesh)'*sin(lambda*xmesh);
end
for ii = 1:size(sol_sep,1)
    sol_sep(ii,:) = sol_sep(ii,:) + xmesh*((T_L - T_0)/L) + T_0;
end

figure(1)
set(surf(tmesh,xmesh,sol_sep'),'linestyle','none');
title(['Separation of variables on bounded domain (first ', num2str(ns), '
terms in series)'])
xlabel('t')
ylabel('x')
zlabel('u(x,t)')
```

Appendix B: Matlab Code for Numerical Solution Plots with pdepe and Heat Dissipation Calculation

```
function BENG221_proj
% diffusion constant
c = 2.91; %j/degC*g
rho = 1.0599 ; %g/cm^3
k = .0132; %W/degC*cm

global D T_B T_L
D = k/(c*rho);
T_L = 15; %degC
T_B = 37; %degC
T_0 = 39; %degC
L = 75; %cm
% domain
dx = 1; % step size in x dimension
dt = 100; % step size in t dimension
xmesh = 0:dx:L; % domain in x
tmesh = 0:dt:10000; % domain in t

% solution using finite differences
nx = length(xmesh); % number of points in x dimension
nt = length(tmesh); % number of points in t dimension

% solution using Matlab's built in "pdepe"
sol_pdepe = pdepe(0,@pdefun,@ic,@bc,xmesh,tmesh);

figure(2)
surf(tmesh,xmesh,sol_pdepe')
title('Normal body temp (37) and device temp (15)')
xlabel('t')
ylabel('x')
zlabel('u(x,t)')
totalValue = 0;
tempdif=1*L*c*rho*5.08^2*pi
for jj=1:nt
    totalValue = 0;
    for ii=1:nx-1
        totalValue=totalValue+dx*(T_B-(sol_pdepe(jj,ii)+sol_pdepe(jj,ii+1))/2)*c*rho*5.08^2*pi;
    end
    if totalValue>tempdif
        tempdif=10000000;
        time2reach=jj
        totalValue2=totalValue
    end
end
end
%heatlost=(totalValue)*c*rho*5.08^2*pi
```

```

heatlost=totalValue

thelost=T_B*L*c*rho*5.08^2*pi

% function definitions for pdepe:
function [c, f, s] = pdefun(x, t, u, DuDx)
% PDE coefficients functions
global D
c = 1;
f = D * DuDx; % diffusion
s = 0; % homogeneous, no driving term

function u0 = ic(x)
% Initial conditions function
% C0=1;
global T_B
u0 = T_B; % heaviside plateau at center

function [pl, ql, pr, qr] = bc(xl, ul, xr, ur, t)
% Boundary conditions function
global T_B T_L
pl = ul-T_B;
ql = 0;
pr = ur-T_L;
qr = 0;

```

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