BENG 186B: Principles of Bioinstrumentation Design

Lecture 2

Linear Circuit Analysis and Dynamical Characteristics

References

Webster, Ch. 1 (Sec. 1.9, 1.10).

The Design and Analysis of Linear Circuits, 6th Ed., Thomas, Rosa and Toussaints, Wiley 2009 (MAE 140 textbook).

http://en.wikipedia.org/wiki/Thévenin%27s_theorem

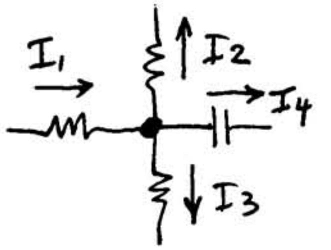
Review of linear circuits

See, e.g.: The Analysis and Design of Linear Circuits, 6th Ed, Thomas, Rosa & Toussaints, Wiley 2009 (MAE 140 book).

- Kirchhoff's current law (KCL): conservation of charge

$$\sum I\text{'s into any node} = 0$$

currents, with sign. Choose the direction of the arrows, and be consistent.

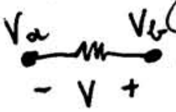


$$: \quad I_1 - I_2 - I_3 - I_4 = 0$$

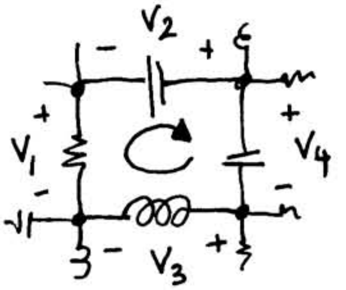
- Kirchhoff's voltage law (KVL): conservation of energy

$$\sum V\text{'s around any closed loop} = 0$$

voltages, with sign. Sometimes in terms of node voltages:



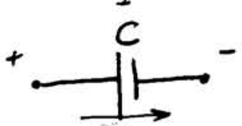
$: V = V_b - V_a$



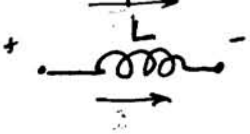
$$V_1 + V_2 - V_4 - V_3 = 0$$

- Circuit elements: relate voltage and current (or their derivatives) → impedance Z

$$V = R \cdot I, \text{ or } Z = R$$



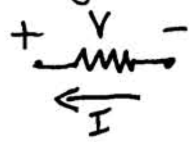
$$I = C \frac{d}{dt} V, \text{ or } Z = \frac{1}{j\omega C}$$



$$V = L \frac{d}{dt} I, \text{ or } Z = j\omega L$$

Note: be consistent with polarities of I and V

e.g.:



$: V = -R \cdot I$

— Independent sources : inputs and supplies

V  : DC voltage (battery, power supply, voltage regulator)

$V(t)$  : AC voltage source (or system input)

typically modeled as sinusoidal: $V = \underset{\text{amplitude}}{A}\, e^{j\overset{\text{phase}}{\varphi}}\, e^{j\overset{\text{radial frequency}}{\omega} t}$

NOTE: V_{pp} : peak to peak voltage $= 2A$

V_{rms} : root mean square (rms) voltage $= \frac{A}{\sqrt{2}}$

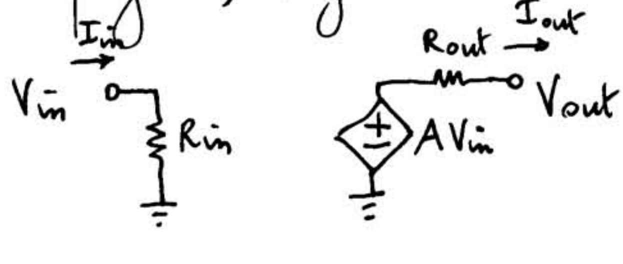
I  : DC current source (bias)

AC current source (system input)

— Dependent sources : voltage or current sources that depend on other voltages or currents in the circuit.

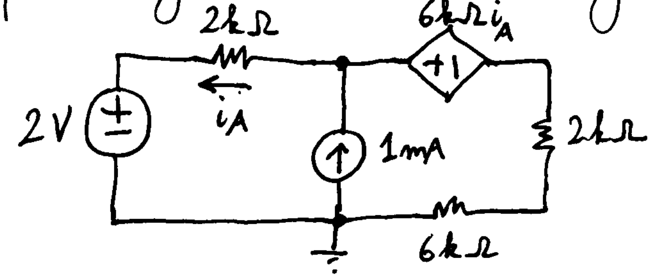
 voltage ;  current

Typically used to represent models of more complex circuits such as amplifiers, e.g.:



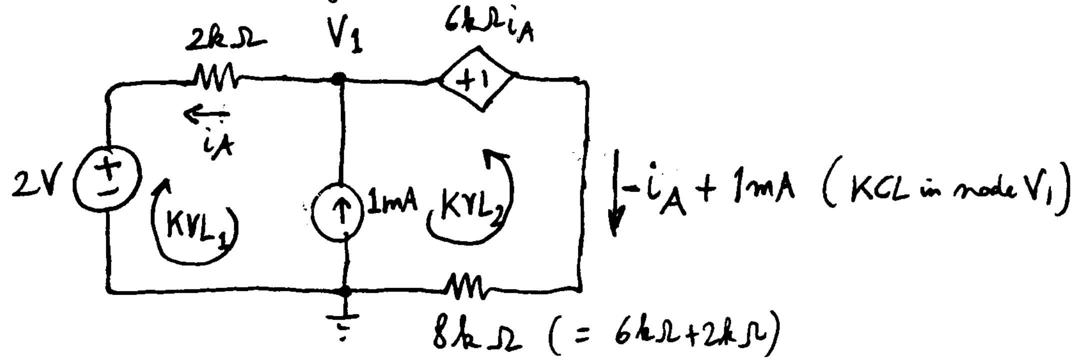
: $V_{out} = AV_{in} - R_{out}I_{out}$

Example: find i_A in the following circuit:



Solution:

Label all unknown voltages & currents, write out known voltages and currents (from KVL & KCL), and combine circuit elements:



$$KVL_1: \quad V_1 = 2V + 2k\Omega \cdot i_A$$

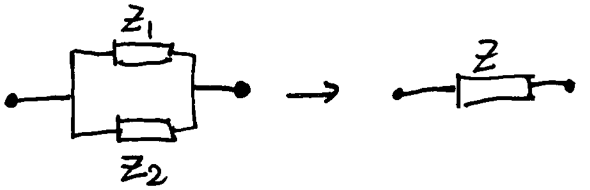
$$KVL_2: \quad V_1 = 8k\Omega\left(-i_A + 1mA\right) + 6k\Omega\, i_A = 8V - 2k\Omega \cdot i_A$$

Eliminating V_1: $\quad 4k\Omega\, i_A = 6V$

$$\Rightarrow \quad i_A = 1.5mA$$

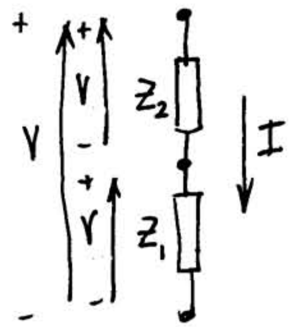
Note: combining circuit elements:

$Z = Z_1 + Z_2$ series combination



$Z = \dfrac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}}$ or $\dfrac{1}{Z} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$ parallel combination

- Series combination:



KCL: $I_1 = I_2 = I$

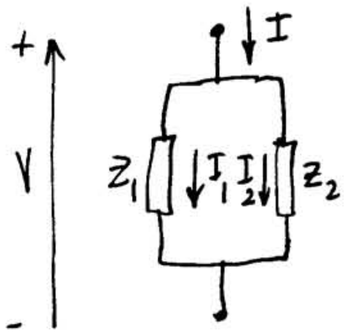
KVL: $V = V_1 + V_2$

$$= Z_1 . I + Z_2 . I$$

$$\Rightarrow Z = \frac{V}{I} = \frac{Z_1 . I + Z_2 . I}{I} = Z_1 + Z_2$$

$\Rightarrow$ IMPEDANCES ADD IN SERIES

- Parallel combination:



KCL: $I = I_1 + I_2$

$$= \frac{V_1}{Z_1} + \frac{V_2}{Z_2}$$

KVL: $V_1 = V_2 = V$

$$\Rightarrow Z = \frac{V}{I} = \frac{V}{\frac{V}{Z_1} + \frac{V}{Z_2}} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 . Z_2}{Z_1 + Z_2}$$

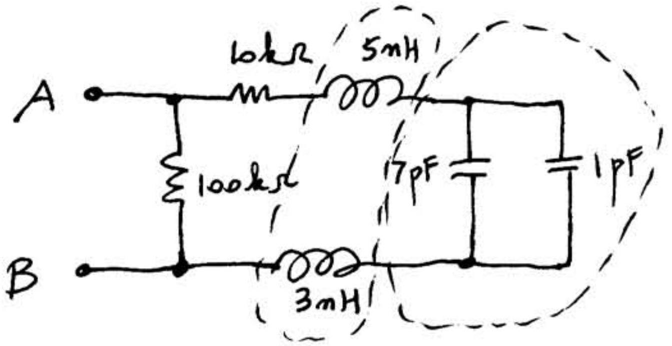
or $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$

or $Y = Y_1 + Y_2$ ADMITTANCE reciprocal of impedance

$\Rightarrow$ ADMITTANCES ADD IN PARALLEL

- Series and parallel combinations can be recursively combined to find impedance of more complex circuits:

<u>Example</u>: find the impedance of the following circuit between nodes A and B:

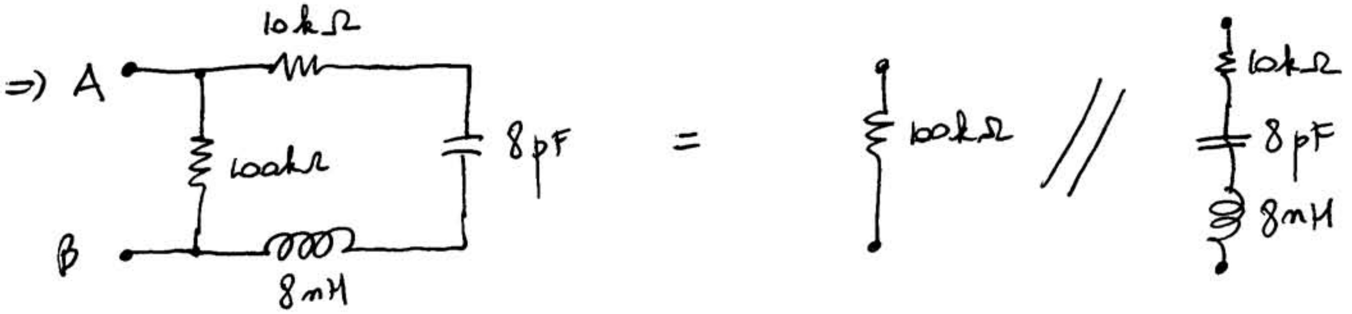


R : $Z = R$ $\Rightarrow$ resistances add in SERIES

L : $Z = j\omega L$ $\Rightarrow$ inductances add in SERIES

C : $Z = \dfrac{1}{j\omega C}$ $\Rightarrow$ capacitances add in PARALLEL

$\Rightarrow$

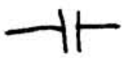


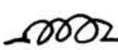
$$\Rightarrow Z = \frac{100\,k\Omega \cdot \left(j\omega\, 8\,mH + 10\,k\Omega + \dfrac{1}{j\omega\, 8\,pF} \right)}{100\,k\Omega + j\omega\, 8\,mH + 10\,k\Omega + \dfrac{1}{j\omega\, 8\,pF}} \quad \begin{matrix} (\times\ j\omega\, 8\,pF) \\ (\times\ j\omega\, 8\,pF) \end{matrix}$$

$$= 100\,k\Omega \cdot \frac{1 + j\omega\, 80\,ns - \omega^2\, 64\, 10^{-21} s^2}{1 + j\omega\, 880\,ns - \omega^2\, 64\, 10^{-21} s^2}$$

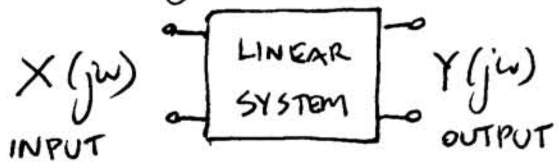
Sanity check: $Z = 100\,k\Omega$ (leftmost branch only)

for $\omega \to 0$ and for $\omega \to \infty$

Indeed:  are open circuit for $\omega \to 0$

 are open circuit for $\omega \to \infty$

Static and dynamical circuits and systems



Transfer function : $\dfrac{Y(j\omega)}{X(j\omega)}$

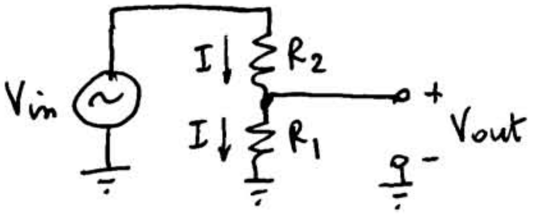
- X and Y can be voltage or current, or other signal types
- $j\omega$ is sometimes written as $D = \frac{d}{dt}$, differential operator for use in the time domain.

— ZEROTH-ORDER INSTRUMENT (static system) :

linear system with constant transfer function : static gain (or sensitivity) independent of frequency.

Circuits with only resistors are always static : the output follows the input instantaneously.

<u>Example</u> : Attenuator implemented as VOLTAGE DIVIDER :



If the output is not loaded, then the same current I flows through R_1 and R_2.

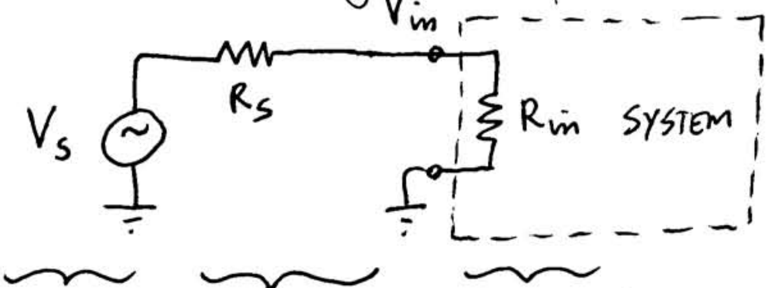
$$\left.\begin{aligned} V_{in} &= R_1 I + R_2 I \quad (\text{KVL}) \\ V_{out} &= R_1 \cdot I \end{aligned}\right\} \Rightarrow V_{out} = \frac{R_1}{R_1+R_2} \cdot V_{in}$$

Attenuation : $\dfrac{V_{out}(j\omega)}{V_{in}(j\omega)} = A_v = \dfrac{R_1}{R_1+R_2}$ where $0 \leq A_v \leq 1$

independent of frequency

Useful where it is desirable to attenuate the amplitude (or range) of a voltage signal for further processing.

NOTE 1: Attenuation due to voltage division is not always intentional in the design, and may be the effect of non-ideal source and system impedances, e.g.:



BIOPOTENTIAL in the body | SKIN AND ELECTRODE IMPEDANCE from body to instrument | INSTRUMENT INPUT IMPEDANCE acquiring the signal

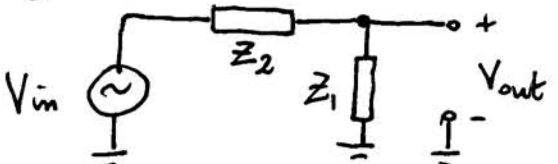
$$\Rightarrow \underbrace{V_{in}}_{\text{signal as measured by the instrument}} = \frac{R_{in}}{R_{in}+R_s} \cdot \underbrace{V_s}_{\text{real signal in the body}} \Rightarrow \text{GOOD DESIGN: } V_{in} \approx V_s$$

for $R_{in} \gg R_s$

→ minimize skin and electrode impedance

→ maximize instrument input impedance

NOTE 2: For frequency-dependent impedances rather than resistors, voltage division may implement frequency-dependent transfer functions:

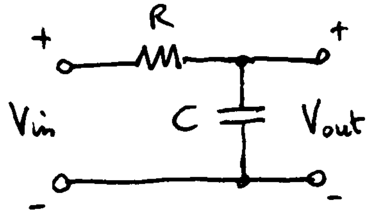


$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{Z_1(j\omega)}{Z_1(j\omega)+Z_2(j\omega)}$$

Dynamical systems can be readily designed by choosing and implementing $Z_1(j\omega)$ and $Z_2(j\omega)$:

– FIRST-ORDER INSTRUMENT : $\dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{b_1 j\omega + b_0}{a_1 j\omega + a_0}$ (single pole)

<u>Example:</u> RC first-order lowpass filter : $\dfrac{V_{out}(j\omega)}{V_{in}(j\omega)} = \dfrac{1}{1 + j\omega\tau}$



$$\left.\begin{aligned} Z_1(j\omega) &= \frac{1}{j\omega C} \\ Z_2(j\omega) &= R \end{aligned}\right\} \Rightarrow \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC}$$

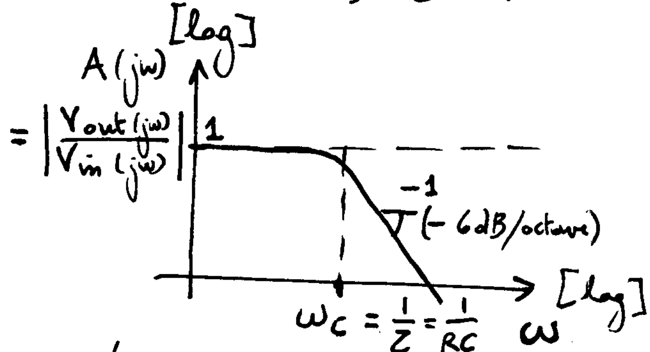
$$\Rightarrow \tau = R.C$$

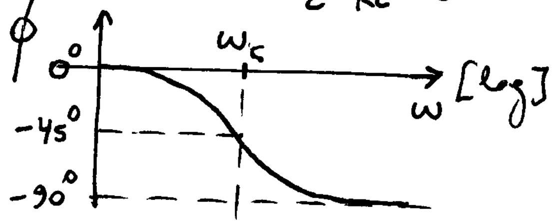
• Frequency response :

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = A(j\omega)\angle\phi(j\omega)$$

$$A(j\omega) = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

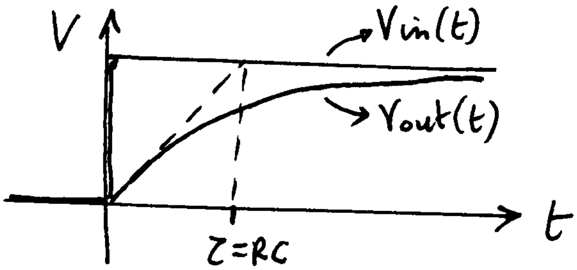
$$\phi(j\omega) = -\arctan(\omega\tau)$$





• Step response :

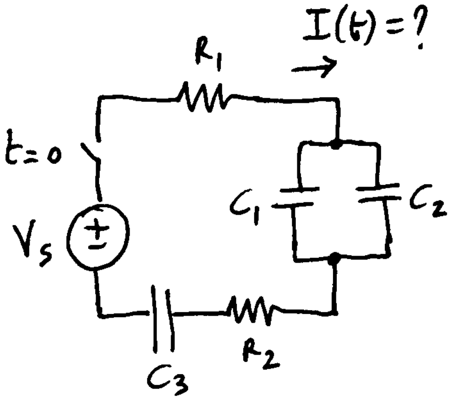
$$V_{out}(t) = 1 - e^{-t/\tau}$$

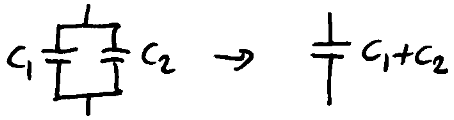


More generally: linear circuits with resistive elements (R's) and with one equivalent dynamical element (L or C), have first order dynamics.

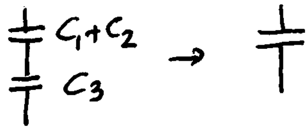
<u>NOTE</u> : combine series or parallel instances of same type (L or C) dynamical elements into single equivalent elements.

Example:



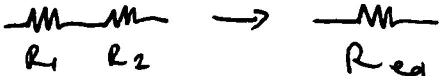


(parallel capacitances add)



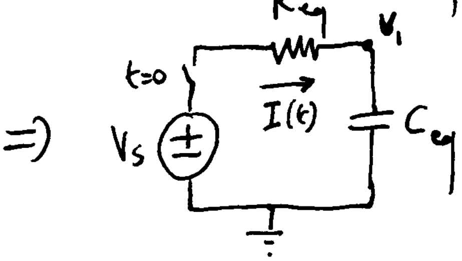
$$C_{eq} = \frac{(C_1+C_2)C_3}{C_1+C_2+C_3}$$

(series capacitances combine reciprocally, additive in $\frac{1}{C}$)



$R_{eq} = R_1 + R_2$

$\Rightarrow$



$$\begin{cases} C_{eq} \dfrac{dV_1}{dt} = I(t) \quad \text{and} \\ V_s = V_1 + R_{eq} \cdot I(t) \quad (\text{for } t \geq 0) \end{cases}$$

2 equations; eliminate V_1 by differentiating the second:

$$0 = \frac{dV_1}{dt} + R_{eq} \cdot \frac{dI}{dt}$$

$$\Rightarrow -\underbrace{R_{eq} \cdot C_{eq}}_{=\tau} \cdot \frac{dI}{dt} = I \quad \Rightarrow \quad I = I_0 \cdot e^{-t/\tau}$$

where I_0 is given by initial conditions.

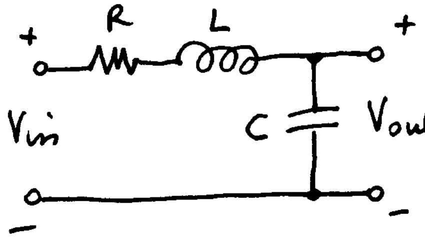
e.g. if $V_1(0) = 0$ (no initial charge on the capacitor)

then $I_0 = V_s / R_{eq}$

– SECOND-ORDER INSTRUMENT: $\dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{b_2(-\omega^2) + b_1 j\omega + b_0}{a_2(-\omega^2) + a_2 j\omega + a_0}$

(two poles, possibly complex)

Example: RLC second-order lowpass: $\dfrac{V_{out}(j\omega)}{V_{in}(j\omega)} = \dfrac{1}{1 + \dfrac{2\zeta j\omega}{\omega_n} - \dfrac{\omega^2}{\omega_n^2}}$



$Z_1(j\omega) = \dfrac{1}{j\omega C}$

$Z_2(j\omega) = R + j\omega L$

$\begin{cases} \omega_n : \text{NATURAL FREQUENCY} \\ \zeta : \text{DAMPING RATIO} \end{cases}$

$$\Rightarrow \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R + j\omega L} = \frac{1}{1 + j\omega RC - \omega^2 LC} \Rightarrow \begin{cases} \omega_n = \dfrac{1}{\sqrt{LC}} \\ \zeta = \dfrac{RC.\omega_n}{2} \end{cases}$$

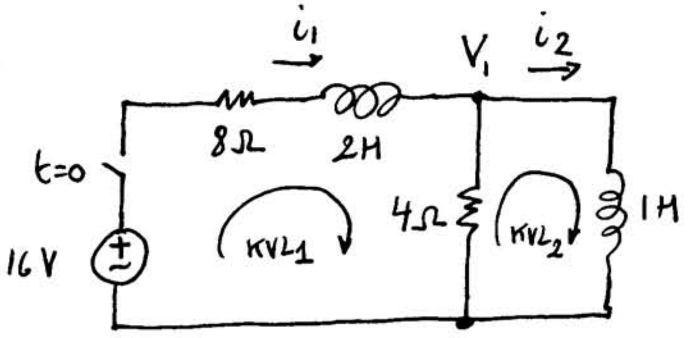
Frequency response and step response: see book for details ...

In essence:

- $\zeta < 1 \Rightarrow$ UNDERDAMPED (two complex poles):
  frequency resonance: peak near $\omega = \omega_n$;
  oscillatory step response
- $\zeta > 1 \Rightarrow$ OVERDAMPED (two real distinct poles):
  no resonance; slow, oscillation-free step response
- $\zeta = 1 \Rightarrow$ CRITICALLY DAMPED (two identical real poles):
  "sweet spot": fastest without resonance/oscillations

More generally: linear circuits with resistive elements (R's) and with TWO equivalent dynamical elements (L or C), have SECOND-ORDER dynamics.

Example:



: find $i_2(t)$.

Solution: use KVL, KCL to get a second-order ODE.

(no sinusoidal input, so use $D = \frac{d}{dt}$ rather than $j\omega$!)

$$\text{KVL}_1 : \quad -16 + 8i_1 + 2\frac{di_1}{dt} + 4\underbrace{(i_1 - i_2)}_{\text{KCL @ } V_1 \text{ node}} = 0 \qquad (1)$$

$$\text{KVL}_2 : \quad 4(i_2 - i_1) + 1.\frac{di_2}{dt} = 0 \qquad (2)$$

Eliminate i_1:

$$\left.\begin{aligned} (2): \quad i_1 &= i_2 + \frac{1}{4}\frac{di_2}{dt} \\ \frac{d}{dt}(2): \quad \frac{di_1}{dt} &= \frac{di_2}{dt} + \frac{1}{4}\frac{d^2 i_2}{dt^2} \end{aligned}\right\} \text{ substitute in } (1)$$

$$\Rightarrow \quad -16 + \left(8i_2 + 2\frac{di_2}{dt}\right) + \left(2\frac{di_2}{dt} + \frac{1}{2}\frac{d^2 i_2}{dt^2}\right) + \frac{di_2}{dt} = 0$$

or

$$\frac{d^2 i_2}{dt^2} + 10\frac{di_2}{dt} + 16\, i_2 = 32.$$

Linear ODE bag of tricks: try a solution of the form:

$$i_2(t) = \underbrace{i_{2\infty}}_{\text{STEADY STATE SOLUTION}} + \underbrace{A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}}_{\text{TRANSIENT; HOMOGENEOUS SOLUTION}}$$

- STEADY STATE; $\frac{d}{dt} = 0$ and $\frac{d^2}{dt^2} = 0 \Rightarrow 16\, i_{2\infty} = 32$ or $i_{2\infty} = 2$
- HOMOGENEOUS SOL.: $\underbrace{\lambda^2 + 10\lambda + 16 = 0}_{\text{characteristic equation}} \Rightarrow$ roots: $\begin{cases} \lambda_1 = -2 \\ \lambda_2 = -8 \end{cases}$

and A_1, A_2 are given by initial conditions of i_2 and $\frac{di_2}{dt}$.

- Steady state:

Often we are not interested in transients, but in the output or behavior of the circuit when it settles into STEADY STATE.

- For sinusoidal inputs: use $j\omega$ rather than $D = \frac{d}{dt}$, no need to solve ODEs!

→ Find transfer function $\frac{Y(j\omega)}{X(j\omega)} = \underbrace{A(\omega)}_{\substack{\text{GAIN} \\ \text{(amplification)}}} \, e^{j\overbrace{\phi(\omega)}^{\text{PHASE}}}$

$$X(t) = \cos(\omega t) \implies Y(t) = A(\omega) . \cos(\omega t + \phi(\omega))$$

in steady state

- For constant inputs (such as voltage/current supplies): same, in the limit $\omega \rightarrow 0$

C : $Z = \frac{1}{j\omega C} \rightarrow \infty \implies$ OPEN CIRCUIT

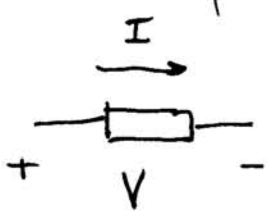
→ remove all capacitors!

L : $Z = j\omega L \rightarrow 0 \implies$ SHORT CIRCUIT

→ all inductors become wires!

- Power dissipation: critical for bioinstrumentation, especially implanted and wireless systems!

- Power dissipated by circuit elements:

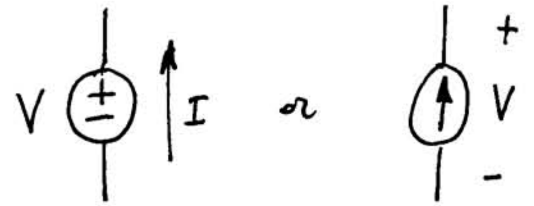


: $P = V.I$

↗ R: $P = RI^2 = \frac{V^2}{R}$

↘ L and C: I and V $90°$ out of phase $\implies$ ZERO AVERAGE POWER

- Power supplied by independent sources:



: $P = V \cdot I$

Note the direction of current for positive power delivered: from lower to higher potential

- Total power consumed by a circuit:

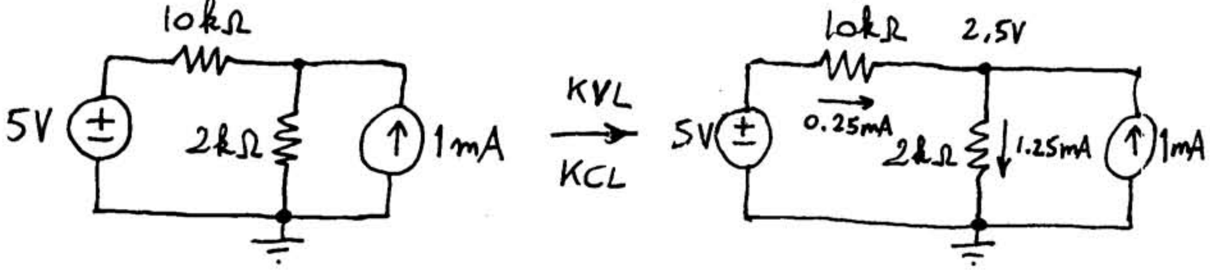
METHOD 1: Sum the power dissipated by all resistors in the circuit.

→ does not account for power consumed by dependent sources, e.g. amplifiers in the circuit

METHOD 2: Sum the power supplied by all independent sources in the circuit, e.g. the power supplies.

→ do not include dependent sources (e.g. amplifiers) in the sum, but include their power supplies (e.g., supply drain of an amplifier)

Example:



METHOD 1: $P_{total} = P_{10k\Omega} + P_{2k\Omega} = 2.5V \times 0.25mA + 2.5V \times 1.25mA = 3.75mW$

METHOD 2: $P_{total} = P_{5V} + P_{1mA} = 5V \times 0.25mA + 2.5V \times 1mA = 3.75mW$ OK!

- AC power, e.g. power drawn from a biosignal source:

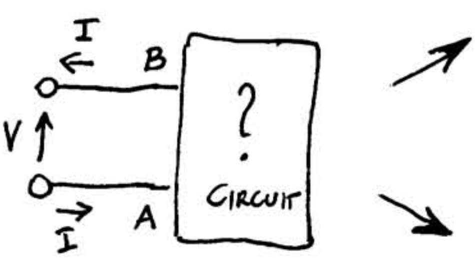
Average power @ ω : $P = \langle V(t) \cdot I(t) \rangle = \frac{1}{2} \text{Real}\left(V(j\omega) \cdot I^*(j\omega)\right)$

(steady state)

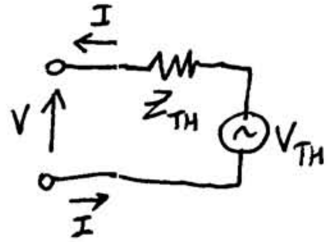
Thévenin and Norton equivalents :

→ useful to find input and output impedance, and transfer function, of a linear system such as an amplifier, filter, etc...

→ reduces the circuit, as observed from a particular port, to a simpler equivalent circuit with just a single source and a single impedance element.

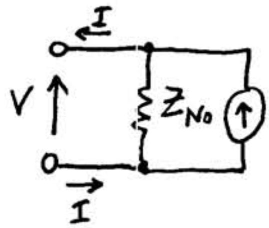


THÉVENIN EQUIVALENT:



: $V = V_{TH} - Z_{TH} \cdot I$

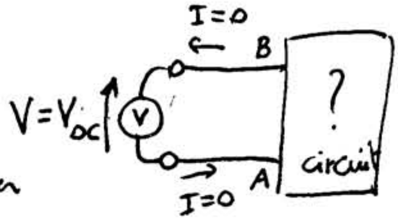
NORTON EQUIVALENT:



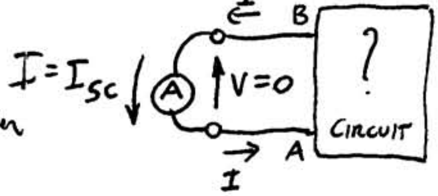
: $I = I_{No} - \dfrac{V}{Z_{No}}$

→ the source and impedance equivalents are found by open-circuit voltage and closed-circuit current measurements (or calculations) over the port:

- $V_{TH} = V_{OC}$ OPEN-CIRCUIT VOLTAGE ; as measured with ideal voltmeter



- $I_{No} = I_{SC}$ SHORT-CIRCUIT CURRENT : as measured with ideal ammeter

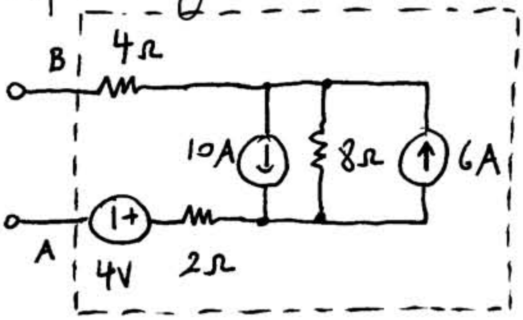


- $Z_{TH} = Z_{N_o} = \dfrac{V_{OC}}{I_{SC}}$ IMPEDANCE :

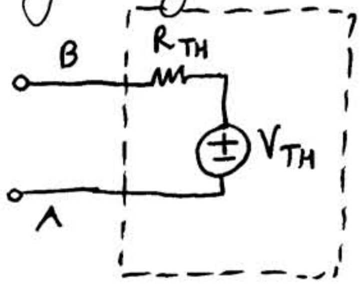
also found as the impedance of the circuit between nodes A and B, by "killing" all internal independent sources:

(±) and (~) → SHORT CIRCUIT (0 V) ; (↓) → OPEN CIRCUIT (0 A)

Example: find the Thévenin equivalent of the following circuit:

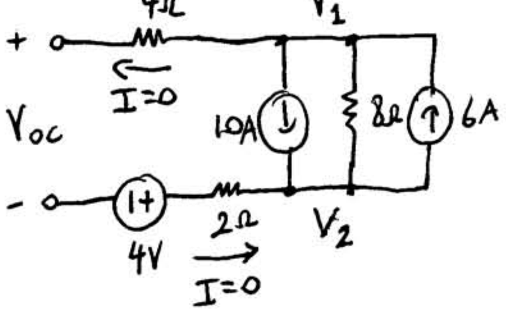


→ from the outside, the same



$R_{TH} = ?$ $V_{TH} = ?$

— $V_{TH} = V_{OC}$: open circuit:



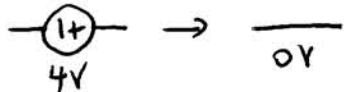
- $I = 0 \Rightarrow$ no voltage drop across 4Ω and 2Ω resistors

$\Rightarrow$ KVL: $V_{OC} = V_1 - V_2 + 4V$

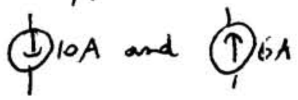
- KCL in V_1 (or V_2): $10A + \frac{V_1 - V_2}{8\Omega} + (-6A) = 0$

$\Rightarrow V_1 - V_2 = -32V \Rightarrow \underline{V_{TH} = V_{OC} = -28V}$

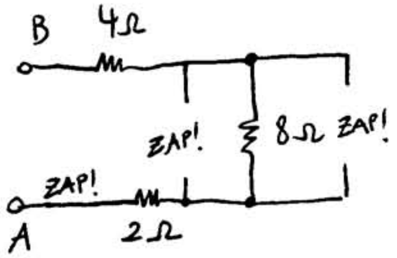
— R_{TH} : "kill" the sources:



(SHORT)



0A (OPEN)



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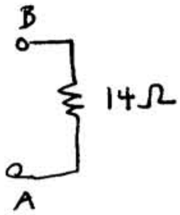


$\underline{R_{TH} = 14\Omega}$

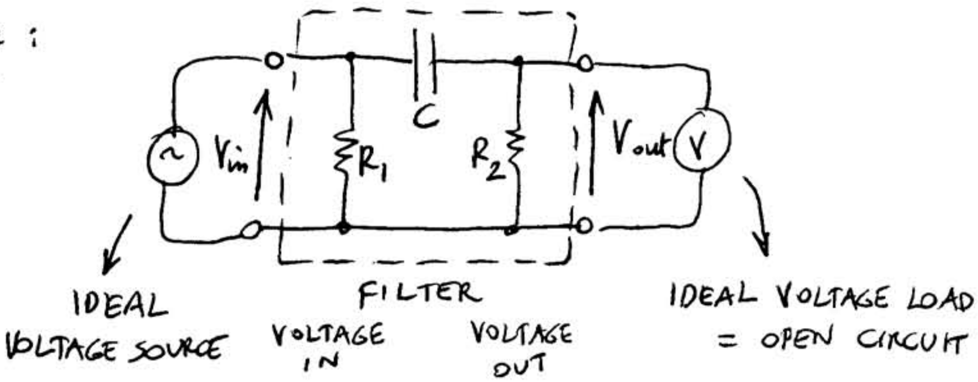
<u>NOTE</u>: only independent sources should be "killed"; dependent sources remain, as they change with current or voltage.

Thévenin / Norton equivalents at input and output ports of a system give the input and output impedance and transfer function of the system.

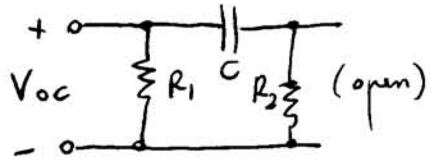
→ The equivalent at the input, for an ideal load at the output, gives the input impedance.

→ The equivalent at the output, for an ideal source at the input, gives the transfer function and output impedance.

Example :

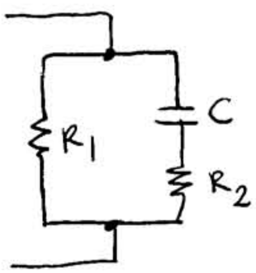


a. INPUT Voltage signal ⇒ THÉVENIN (Current ⇒ NORTON)



No sources ⇒ $V_{oc} = 0$ (OK!)

Z_{TH} :

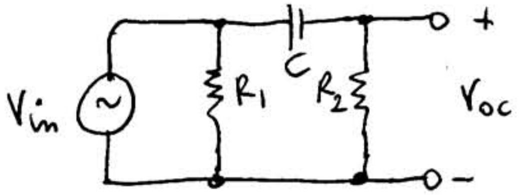


parallel-series

$$Z_{TH} = \frac{R_1\left(R_2 + \frac{1}{jwC}\right)}{R_1 + R_2 + \frac{1}{jwC}} = R_1 \cdot \frac{1 + jwR_2C}{1 + jw(R_1+R_2)C}$$

$$\Rightarrow Z_{in} = R_1 \cdot \frac{1 + jwR_2C}{1 + jw(R_1+R_2)C}$$

6. OUTPUT   Voltage signal $\Rightarrow$ THÉVENIN



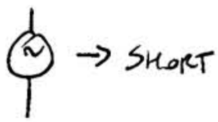
R_1 doesn't matter as it is driven by V_{in} !

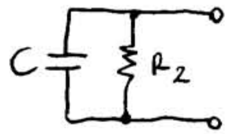
Voltage divider: $$V_{oc} = \frac{R_2}{R_2 + \frac{1}{j\omega C}} \cdot V_{in} = \frac{j\omega R_2 C}{1 + j\omega R_2 C} \cdot V_{in}$$

$$\Rightarrow V_{TH} = A_v(j\omega) \cdot V_{in}, \text{ where } A_v(j\omega) = \frac{j\omega R_2 C}{1 + j\omega R_2 C}$$

is the TRANSFER FUNCTION $\dfrac{V_{out}(j\omega)}{V_{in}(j\omega)}$

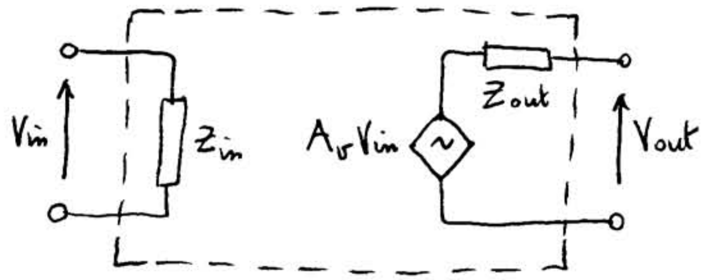
Z_{TH} :





$$Z_{TH} = \frac{\frac{1}{j\omega C} \cdot R_2}{\frac{1}{j\omega C} + R_2} = \frac{R_2}{1 + j\omega R_2 C} = Z_{out}$$

$\Rightarrow$



$$Z_{in}(j\omega) = R_1 \cdot \frac{1 + j\omega R_2 C}{1 + j\omega (R_1 + R_2) C} \qquad A_v(j\omega) = \frac{j\omega R_2 C}{1 + j\omega R_2 C} \qquad Z_{out}(j\omega) = \frac{R_2}{1 + j\omega R_2 C}$$

Exercise : How do transfer function and output impedance change if input is CURRENT rather than VOLTAGE ?