Hypothermia and generation/loss of body heat in tissue

Consider the temperature distribution \( u(x,t) \) in body tissue due to exposure to extreme cold. The skin at \( x = 0 \) is in direct contact with ice and freezing water at temperature \( T_0 = 0\,^\circ\text{C} \). Bone at depth \( x = L \) behind the tissue is thermally insulating (does not conduct heat). The layer of tissue with thickness \( L \) has uniform \((0 \leq x \leq L)\) mass density \( \rho \), heat capacitance \( c \), and heat conductivity \( K_0 \). Initially the tissue is at nominal body temperature \( T_b = 37\,^\circ\text{C} \). Heat is being generated and supplied by vasculature at a constant rate with non-uniform distribution \( Q(x) \) in the tissue.

a (10 pts)

Draw a model of the tissue, specifying the boundary conditions. Write out the partial differential equation governing the diffusion of heat in the tissue, including boundary and initial conditions. Express the heat diffusivity \( D \) in terms of the given physical constants.

b (20 pts)

Assume first that the heat generation source term is zero \( (Q = 0) \). Express the general solution \( u_H(x,t) \) to this homogeneous problem, by separation of variables, as a series expansion of eigenmodes \( u_k(x,t) \).

c (10 pts)

Now consider the initial conditions in addition to the boundary conditions, still without heat generation. Write down the resulting solution to the temperature distribution \( u(x,t) \).

d (20 pts)

Now assume that the vasculature generates heat at constant rate with profile \( Q(x) = Q_0 \sin(\pi x / L) \) in the tissue. Find a particular solution \( u_P(x) \), for the given boundary conditions, at equilibrium \( (\partial / \partial t \equiv 0) \).

e (20 pts)

Show that the general solution \( u(x,t) \) with the given heat generation and boundary conditions is given by the sum of the particular solution \( u_P(x) \) and the general solution to the homogeneous problem \( u_H(x,t) \). Solve for \( u(x,t) \) with the given initial conditions.

f (20 pts)

How much heat \( Q_0 \) must the body generate in order for the tissue at depth \( L \) to remain at all times within \( \Delta T = 5^\circ\text{C} \) of the nominal body temperature \( T_b \)? If the body is unable to generate sufficient heat, approximately how much time does it take for the tissue temperature at depth \( L \) to drop \( \Delta T \) below \( T_b \)? Hint: Consider the relatively rapid decay of the higher eigenmodes in the series expansion of the homogeneous solution, which lets you discard all but the first eigenmode \( u_1(x,t) \) as a first-order approximation.