Diffusion in 3-D Cartesian Space

This homework is an exercise in 3-D diffusion, extending the analytical techniques that you learned for solving the diffusion equation to 3-D Cartesian coordinates.

Consider an antacid capsule in the stomach of an acid reflux patient, releasing antacid in the surrounding gastric fluid. The capsule is not quite the oval shape of pills that you normally find over the counter in pharmacies, not because we don’t like ovals, but because we can’t deal with them yet, for now, in this mathematical methods class. So to keep things simple in the analysis we assume that the capsule is shaped as a “hyperrectangle,” or “cuboid,” with width/depth $W = 5$ mm in dimensions $x$ and $y$ (i.e., $0 \leq x, y \leq W$), and with length $L = 2$ cm in dimension $z$ (i.e., $0 \leq z \leq L$). The capsule has fully insulating boundaries on five faces of the capsule for $x = 0$ (left face), $x = W$ (right face), $y = 0$ (front face), $y = W$ (rear face), and $z = 0$ (bottom face). The top face $z = L$ is completely open in contact with the gastric fluid of the stomach which completely absorbs the antacid upon contact. Initially at time $t = 0$ a quantity $G_0 = 1$ µmol of antacid, with other inactive ingredients, is uniformly distributed in the capsule. The diffusivity of antacid in the capsule is $D = 1$ mm²/s.

a (10 pts)
Write the partial differential equation governing the antacid concentration $u(x, y, z, t)$ in the capsule, including the initial and boundary conditions.

b (25 pts)
Solve for the general solution by separation of variables, as a series of eigenmodes.

c (15 pts)
Now find the solution for the given initial conditions. Does the solution depend on all the coordinates $x, y, z,$ and $t$, and why?

d (35 pts)
Derive and graph the total quantity $R(t)$ of antacid released by the capsule into the stomach, as a function of time $t$. Verify that $R(t)$ is 0 for $t = 0$, and $G_0$ for $t \to \infty$. How much time does it approximately take for 99% of the total antacid $G_0$ to be released into the stomach? Hint: That takes a while, and by then most of the eigenmodes except for the first have all but vanished, so you can compute this time by hand rather than guesstimating it from eyeballing the graph.

e (15 pts)
Now consider that the capsule is active rather than passive through a self-sustaining reaction, generating antacid continuously over time and uniformly across the capsule at a constant rate $f = 1$ µmol s⁻¹ cm⁻³. Find the equilibrium rate of release of antacid into the stomach.