BENG 221
Mathematical Methods in Bioengineering

Lecture 1
Introduction
ODEs and Linear Systems

Gert Cauwenberghs
Department of Bioengineering
UC San Diego
Course Objectives

1. Acquire methods for quantitative analysis and prediction of biophysical processes involving spatial and temporal dynamics:
   - Derive partial differential equations from physical principles;
   - Formulate boundary conditions from physical and operational constraints;
   - Use engineering mathematical tools of linear systems analysis to find a solution or a class of solutions;

2. Learn to apply these methods to solve engineering problems in medicine and biology:
   - Formulate a bioengineering problem in quantitative terms;
   - Simplify (linearize) the problem where warranted;
   - Solve the problem, interpret the results, and draw conclusions to guide further design.

3. Enjoy!
Today’s Coverage:

Ordinary Differential Equations

Linear Time-Invariant Systems

Eigenmodes

Convolution and Response Functions
ODE Problem Formulation

Solve for the dynamics of $n$ variables $x_1(t), x_2(t), \ldots x_n(t)$ in time (or other ordinate) $t$ described by $m$ differential equations:

\[
F_i \left( x_1, \frac{dx_1}{dt}, \cdots \frac{d^k x_1}{dt^k}, \cdots \right.
\]
\[
\left. x_2, \frac{dx_2}{dt}, \cdots \frac{d^k x_2}{dt^k}, \cdots \right.
\]
\[
\left. x_n, \frac{dx_n}{dt}, \cdots \frac{d^k x_n}{dt^k} \right) = 0
\]

for $i = 1, \ldots m$, where $m \leq n$ and $k \leq n$. Solutions are generally not unique. A unique solution, or a reduced set of solutions, is determined by specifying initial or boundary conditions on the variables.
ODE Examples

Kinetics of mass $m$ with potential $V(x)$:

$$\frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + V(x) = 0 \quad (2)$$

Two masses with coupled potential $V(x)$:

$$\frac{1}{2} m_1 \left( \frac{dx_1}{dt} \right)^2 + \frac{1}{2} m_2 \left( \frac{dx_2}{dt} \right)^2 + V(x_1, x_2) = 0 \quad (3)$$

Second order nonlinear ODE:

$$x \frac{d^2 x}{dt^2} = \frac{1}{2} \left( \frac{dx}{dt} \right)^2 \quad (4)$$
ODE in Canonical Form

In *canonical form*, a set of $n$ ODEs specify the first order derivatives of each of $n$ single variables in the other variables, without coupling between derivatives or to higher order derivatives:

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots x_n) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots x_n) \\
\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots x_n).
\end{align*}
\]  

(5)

Not every system of ODEs can be formulated in canonical form. An important class of ODEs that can be formulated in canonical form are *linear ODEs*. 

Amplitude stabilized quadrature oscillator:

\[
\begin{align*}
\frac{dx}{dt} &= -y - (x^2 + y^2 - 1) x \\
\frac{dy}{dt} &= x - (x^2 + y^2 - 1) y
\end{align*}
\]  

(6)

Any first-order canonical ODE without explicit time dependence can be solved by separation of variables, e.g.,

\[
\frac{dx}{dt} = \frac{1 + x^2}{x}
\]  

(7)
Initial and Boundary Conditions

Initial conditions are values for the variables, and some of their derivatives of various order, specified at one initial point in time $t_0$, e.g., $t = 0$:

$$\frac{d^i x_j}{dt^i}(0) = c_{ij}, \quad i = 0, \ldots, m, \quad j = 1, \ldots, n. \quad (8)$$

Boundary conditions are more general conditions linking the variables, and/or their first and higher derivatives, at one or several points in time $t_k$:

$$g_l(\ldots, \frac{d^i x_j}{dt^i}(t_k), \ldots) = 0. \quad (9)$$
ICs in Canonical Form

For ODEs in canonical form, initial conditions for each of the variables are specified at initial time $t_0$, e.g., $t = 0$:

**Canonical IC**

\[
\begin{align*}
    x_1(0) &= c_1 \\
    x_2(0) &= c_2 \\
    \vdots \\
    x_n(0) &= c_n
\end{align*}
\]  

(10)

ICs for first or higher order derivatives are not required for canonical ODEs.
Linear Canonical ODEs

Linear time-invariant (LTI) systems can be described by linear canonical ODEs with constant coefficients:

**LTI ODE**

\[
\frac{dx}{dt} = Ax + b \tag{11}
\]

with \( x = (x_1, \ldots, x_n)^T \), and with linear initial conditions:

**LTI IC**

\[
x(0) = e \tag{12}
\]

or linear boundary conditions at two, or more generally several, time points:

**LTI BC**

\[
Cx(0) + Dx(T) = e \tag{13}
\]
Examples abound in biomechanical and electromechanical systems (including cardiovascular system, and MEMS biosensors), and more recently bioinformatics and systems biology.

A classic example is the harmonic oscillator \((k = 0)\), and more generally the damped oscillator or resonator:

\[
\begin{cases}
   \frac{du}{dt} = v \\
   m\frac{dv}{dt} = -k\ u - \gamma\ v + f_{\text{ext}}
\end{cases}
\]  

where \(u\) represents some physical form of deflection, and \(v\) its velocity. Typical parameters include mass/inertia \(m\), stiffness \(k\), and friction \(\gamma\). The inhomogeneous term \(f_{\text{ext}}\) represents an external force acting on the resonator.
LTI Homogeneous ODEs

In general, LTI ODEs are *inhomogeneous*. *Homogeneous* LTI ODEs are those for which $x \equiv 0$ is a valid solution. This is the case for LTI ODEs with zero driving force $b = 0$ and zero IC/BC:

**LTI Homogeneous ODE**

$$\frac{dx}{dt} = Ax \quad (15)$$

**LTI Homogeneous IC**

$$Cx(0) = 0 \quad (16)$$

**LTI Homogeneous BC**

$$Cx(0) + Dx(T) = 0. \quad (17)$$

*Eigenmodes*, arbitrarily scaled non-trivial solutions $x \neq 0$, exist for under-determined IC/BC (rank-deficient $C$ and $D$).
Eigenmode Analysis

Eigenvalue-eigenvector decomposition of the matrix $A$ yields the eigenmodes of LTI homogeneous ODEs. Let:

\[ A \mathbf{x}_i = \lambda_i \mathbf{x}_i \]  

(18)

with eigenvectors $\mathbf{x}_i$ and corresponding eigenvalues $\lambda_i$. Then

**Eigenmodes**

\[ \mathbf{x}(t) = c_i \mathbf{x}_i e^{\lambda_i t} \]  

(19)

are *eigenmode* solutions to the LTI homogeneous ODEs (15) for any scalars $c_i$. There are $n$ such eigenmodes, where $n$ is the rank of $A$ (typically, the number of LTI homogeneous ODEs).
Orthonormality and Inhomogeneous IC/BCs

The general solution is expressed as a linear combination of eigenmodes:

\[ x(t) = \sum_{i=1}^{n} c_i \ x_i \ e^{\lambda_i t} \]  

(20)

For symmetric matrix \( A \) \((A_{ij} = A_{ji})\) the set of eigenvectors \( x_i \) is orthonormal:

\[ x_i^T x_j = \delta_{ij} \]  

(21)

so that the solution to the homogeneous ODEs (15) with inhomogeneous ICs (12) reduces to \( c_i = x_i^T x(0) \), or:

**LTI inhomogenous IC solution (symmetric \( A \))**

\[ x(t) = \sum_{i=1}^{n} x_i^T x(0) \ x_i \ e^{\lambda_i t} \]  

(22)
Superposition and Time-Invariance

Linear time-invariant (LTI) homogeneous ODE systems satisfy the following useful properties:

**LTI ODE**

1. **Superposition:** If \( x(t) \) and \( y(t) \) are solutions, then \( A \ x(t) + B \ y(t) \) must also be solutions for any constant \( A \) and \( B \).

2. **Time Invariance:** If \( x(t) \) is a solution, then so is \( x(t + \Delta t) \) for any time displacement \( \Delta t \).

An important consequence is that solutions to LTI inhomogeneous ODEs are readily obtained from solutions to the homogeneous problem through *convolution*. This observation is the basis for extensive use of the *Laplace and Fourier transforms* to study and solve LTI problems in engineering.
Impulse Response and Convolution

Let $h(t)$ the *impulse response* of a LTI system to a delta Dirac function at time zero:

$$\frac{dh}{dt} = \mathcal{L}(h) + \delta(t) \quad (23)$$

then, owing to the principle of superposition and time invariance, the response $u(t)$ to an arbitrary stimulus over time $f(t)$

$$\frac{du}{dt} = \mathcal{L}(u) + f(t) \quad (24)$$

is given by:

**Convolution**

$$u(t) = \int_{-\infty}^{+\infty} f(\theta) \ h(t - \theta) \ d\theta. \quad (25)$$
Fourier Transfer Function

Linear convolution in the time domain (25)

\[ u(t) = \int_{-\infty}^{+\infty} f(\theta) h(t - \theta) \, d\theta \]

transforms to a linear product in the Fourier domain:

\[ U(j\omega) = F(j\omega) \, H(j\omega) \] (26)

where

\[ U(j\omega) = \mathcal{F}(u(t)) = \int_{-\infty}^{+\infty} u(\theta) \, e^{-j\omega \theta} \, d\theta \] (27)

is the Fourier transform of \( u \).

The transfer function \( H(j\omega) \) is the Fourier transform of the impulse response \( h(t) \).
Laplace Transfer Function

For *causal systems*

\[ h(t) \equiv 0 \quad \text{for} \quad t < 0 \quad (28) \]

the identical product form (26)

\[ U(s) = F(s) \cdot H(s) \quad (29) \]

holds also for the Laplace transform

\[ U(s) = \mathcal{L}(u(t)) = \int_{0}^{+\infty} u(\theta) \cdot e^{-s\theta} \, d\theta \quad (30) \]

where \( s = j\omega \).
Bibliography