Lecture 10

General Solutions to Inhomogeneous PDEs using Green's Functions

References

See also Lecture 7 notes. Haberman APDE, Sec. 11.3. http://en.wikipedia.org/wiki/Green's_function 3) INHOMOGENEOUS PDES WITH INHOMOGENEOUS B.C.

For now, assume any
$$M_{I}(X,t)$$
 that satisfies the B.C.,
as the Green's solution does not depend on it.
$$\begin{cases} M_{I}(0,t) = M_{0}(t) \text{ B.C.} \\ M_{I}(X,0) = 0 \text{ J.C.} \\ M_{I}(L,t) = M_{L}(t) \text{ B.C.} \end{cases}$$

Homogeneous B.C. PDE for
$$\mathcal{M}_{H}(x,t)$$
:

$$\frac{\partial \mathcal{M}_{H}}{\partial t} = k \frac{\partial^{2} \mathcal{M}_{H}}{\partial x^{2}} + Q_{H}(x,t) \quad \text{with} \begin{cases} \mathcal{M}(0,t) = 0 \ (\text{VALUE}) \\ \mathcal{M}(L,t) = 0 \ (\text{VALUE}) \\ \mathcal{M}(X,0) = g(X) \\ \mathcal{M}(X,0) = g(X) \end{cases}$$

Substitution of (1) in the inhomogeneous PDE gives:

$$Q(x,t) = Q_H(x,t) + \bigcup_{cot}^{0} M_I(x,t) - k \bigcup_{cot}^{0^2} M_I(x,t)$$

on $Q_{H} = Q - \left(\frac{\partial}{\partial k} - k \frac{\partial^{2}}{\partial x^{2}}\right) M_{I}$

Substitution of the homogeneous solution using Green's functions:

$$\begin{aligned}
\mu(x,t) &= & M_{H}(x,t) + & M_{I}(x,t) \\
&= & \int_{g}(x_{0})G(x,t;x_{0},0)dx_{0} + \int_{G}^{t}G_{H}(x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0}dt_{0} \\
&= & \int_{G}^{t}g(x_{0})G(x,t;x_{0},0)dx_{0} + \int_{G}^{t}G(x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0}dt_{0} \\
&= & \int_{G}^{t}g(x_{0})G(x,t;x_{0},0)dx_{0} + \int_{G}^{t}G(x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0}dt_{0} \\
&= & \int_{G}^{t}g(x_{0})G(x,t;x_{0},0)dx_{0} + \int_{G}^{t}G(x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0}dt_{0} \\
&= & \int_{G}^{t}g(x_{0})G(x,t;x_{0},t_{0})dx_{0} \\
&= & \int_{G}^{t}g(x_{0})G(x,t;x_{0},t_{0})G(x,t;x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0}dt_{0} \\
&= & \int_{G}^{t}g(x_{0},t_{0})G(x,t;x_{0},t_{0})\int_{G}^{t}g(x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0} \\
&= & \int_{G}^{t}\pi_{I}(x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0} \\
&= & \int_{G}^{t}\pi_{I}(x_{0},t_{0})G(x,t;x_{0},t_{0})dx_{0}$$

Integration by parts, twice in X₀:

$$\int_{X} = \int_{0}^{L} \frac{\partial^{2}}{\partial X_{0}^{2}} \mathcal{M}_{I}(X_{0}, t_{0}) G(X_{1}t_{j} X_{0}, t_{0}) dX_{0}$$

$$= \left[\frac{\partial}{\partial X_{0}} \mathcal{M}_{I}(X_{0}, \cdot) G(\cdot, \cdot; X_{0}, \cdot) \right]_{0}^{L} - \int_{0}^{L} \frac{\partial}{\partial X_{0}} \mathcal{M}_{I}(X_{0}, \cdot) \frac{\partial}{\partial X_{0}} G(\cdot, \cdot; X_{0}, \cdot) dX_{0}$$

$$= - \left[\mathcal{M}_{I}(X_{0}, \cdot) \frac{\partial}{\partial X_{0}} G(\cdot, \cdot; X_{0}, \cdot) \right]_{0}^{L} + \int_{0}^{L} \mathcal{M}_{I}(X_{0}, \cdot) \frac{\partial^{2}}{\partial X_{0}^{2}} G(\cdot, \cdot; X_{0}, \cdot) dX_{0}$$

$$=) (*) = (x) + (T) + M_{L}(x,t)$$

$$= -k \int \left[M_{I}(x_{0},t_{0}) \frac{\partial}{\partial x_{0}} G(x,t;x_{0},t_{0}) \right]_{0}^{L} dt_{0}$$

$$= \int M_{I}(x_{0},t_{0}) \cdot \left(\frac{\partial}{\partial x_{0}} + \frac{\partial}{\partial x_{0}} \frac{\partial}{\partial x_{0}} G(x,t;x_{0},t_{0}) \right]_{0}^{L} dx_{0} dt_{0}$$

$$= 0 \quad breause \quad d \quad ReciProcity$$

$$= \int M_{0}(t_{0}) \quad k \frac{\partial}{\partial x_{0}} G(x,t;0,t_{0}) dt_{0} - \int M_{L}(t_{0}) \quad k \frac{\partial}{\partial x_{0}} G(x,t;L,t_{0}) dt_{0}$$

$$= \int B.c. @ x=0 \qquad B.c. @ x=L$$

independent of chosen form of MI (Xo, to)

$$= \int \mathcal{M}(x,t) = \int g(x_0)G(x,t;x_0,0)dx_0 + \iint Q(x_0,t_0)G(x,t;x_0,t_0)dx_0dt_0$$

$$= \int g(x_0)G(x,t;x_0,0)dx_0 + \iint Q(x_0,t_0)G(x,t;x_0,t_0)dx_0dt_0$$

$$= \int \mathcal{M}_0(t_0) \frac{1}{k_{theree}} \frac{1}{Q(x_0,t_0)}G(x,t;0,t_0)dt_0 - \int \mathcal{M}_1(t_0) \frac{1}{k_{theree}} \frac{1}{Q(x_0,t_0)}G(x,t;1,t_0)dt_0$$

$$= \int \mathcal{M}_0(t_0) \frac{1}{k_{theree}} \frac{1}{Q(x_0,t_0)}G(x,t;0,t_0)dt_0 - \int \mathcal{M}_1(t_0) \frac{1}{k_{theree}} \frac{1}{Q(x_0,t_0)}G(x,t;1,t_0)dt_0$$

$$= \int \mathcal{M}_0(t_0) \frac{1}{k_{theree}} \frac{1}{Q(x_0,t_0)}G(x,t;1,t_0)dt_0$$

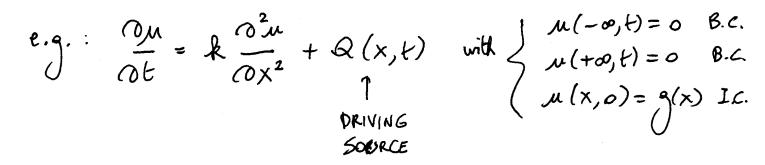
$$= \int \mathcal{M}_0(t_0) \frac{1}{Q(x_0,t_0)}G(x,t;0,t_0)dt_0 + \int \mathcal{M}_1(t_0) \frac{1}{Q(x_0,t_0)}G(x,t;1,t_0)dt_0$$

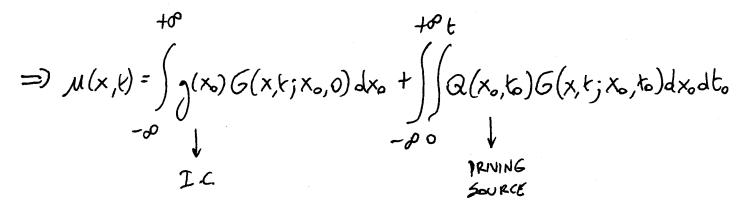
$$= \int \mathcal{M}_0(t_0) \frac{1}{Q(x_0,t_0)}G(x,t;0,t_0)dt_0 + \int \mathcal{M}_1(t_0) \frac{1}{Q(x_0,t_0)}G(x,t;1,t_0)dt_0$$

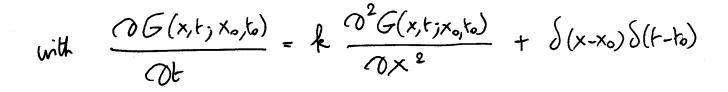
$$= \int \mathcal{M}_0(t_0) \frac{1}{Q(x,t;0,t_0)}dt_0 + \int \mathcal{M}_1(t_0) \frac{1}{Q(x,t;1,t_0)}dt_0$$

$$= \int \mathcal{M}_0(t_0) \frac{1}{Q(x,t;0,t_0)}dt_0$$

GREEN'S FUNCTION FOR PDES ON UNBOUNDED POMAINS







Solution using Laplace in time, and Fourier in space :

 $G(x,t;x_{o},t_{o}) = \frac{1}{2\sqrt{TR(t-t_{o})}} \cdot e^{-\frac{(x-x_{o})^{2}}{4k(t-t_{o})}}$

Generalized to 3 dimensions (using Guen's Theorem.):
• HEAT/DIFFUSION EQ. :
$$\int_{CU}^{0} H = \frac{1}{\sqrt{2}} \int_{C}^{2} H + Q(\vec{x},t)$$

 $(\eta \cdot 523 - 52)$
 $M(\vec{x},t) = \int_{C}^{1} \iint_{C} G(\vec{x},t;\vec{x}_{0},t_{0}) Q(\vec{x}_{0},t_{0}) \frac{1}{\sqrt{2}} \int_{C}^{2} H(\vec{x}_{0},t_{0}) - H(\vec{x}_{0},t_{0}) \frac{1}{\sqrt{2}} \int_{C}^{2} (\vec{x},t;\vec{x}_{0},t_{0}) \frac{1}{\sqrt{2}} \int_{C}^{2} (\vec{x},t;\vec{x},t;\vec{x}_{0},t_{0}) \frac{1}{\sqrt{2}} \int_{C}^{2} (\vec{x},t;\vec{x},t;\vec{x}_{0},t_{0}) \frac{1}{\sqrt{2}} \int_{C}^{2} (\vec{x},t;\vec{x},t;\vec{x}_$