BEGN 221: Mathematical Methods in Bioengineering

Lecture 13

Review of Vector Calculus

References

H.M. Schey, *Div grad curl and all that: an informal text on vector calculus*, 4th ed., New York: W.W. Norton, 2005.

http://en.wikipedia.org/wiki/Gradient

http://en.wikipedia.org/wiki/Gradient_theorem

http://en.wikipedia.org/wiki/Divergence

http://en.wikipedia.org/wiki/Divergence_theorem

http://en.wikipedia.org/wiki/Curl_(mathematics)

http://en.wikipedia.org/wiki/Stokes%27_theorem

http://en.wikipedia.org/wiki/Spherical_coordinates

http://mathworld.wolfram.com/SphericalCoordinates.html

GRADIENT:

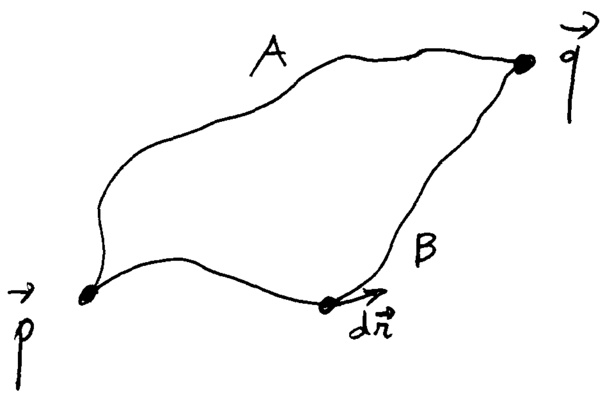
$$\vec{F} = \vec{\nabla} u \quad \text{with} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$$

$$\begin{cases} F_x = \dfrac{\partial u}{\partial x} \\ F_y = \dfrac{\partial u}{\partial y} \\ F_z = \dfrac{\partial u}{\partial z} \end{cases}$$

$\vec{F}$ points in direction of steepest ascent of $u(x,y,z)$

$\vec{F}$: "FIELD" (conservative vector field)

u : "POTENTIAL"



$$u(\vec{q}) - u(\vec{p}) = \int_A \vec{F} \cdot d\vec{r} = \int_B \vec{F} \cdot d\vec{r}$$

$$\vec{F} \cdot d\vec{r} = \vec{\nabla} u \cdot d\vec{r}$$

$$= \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z} \cdot dz$$

$$= du\,(d\vec{r})$$

- du is largest for $d\vec{r} \parallel \vec{\nabla} u$

  ($du = |\vec{\nabla} u| \cdot |d\vec{r}| \cdot \cos\alpha$ = max for $\cos\alpha = 1$, $\alpha = 0$)

  $\Rightarrow$ $\vec{\nabla} u$ points in direction of STEEPEST ASCENT

- $$\int_{\vec{p}}^{\vec{q}} \vec{F} \cdot d\vec{r} = \int_{\vec{p}}^{\vec{q}} \vec{\nabla} u \cdot d\vec{r} = \int_{u(\vec{p})}^{u(\vec{q})} du = u(\vec{q}) - u(\vec{p})$$

Examples:

- Fick's law of diffusion:

$$\underset{\text{FLOW}}{\vec{j}} = - \underset{\text{DIFFUSIVITY}}{D} \; \underset{\text{GRADIENT IN CONCENTRATION}}{\vec{\nabla} u(\vec{r},t)}$$

$$\left[\frac{mol}{m^2 s}\right] \qquad \left[\frac{m^2}{s}\right] \quad \left[\frac{1}{m}\right]\left[\frac{mol}{m^3}\right]$$

- Fourier's law of heat conduction:

$$\underset{\text{HEAT FLOW}}{\vec{j}_e} = - \underset{\text{THERMAL CONDUCTIVITY}}{K_0} \; \underset{\text{GRADIENT IN TEMPERATURE}}{\vec{\nabla} u(\vec{r},t)}$$

$$\left[\frac{J}{m^2 s}\right] \qquad \left[\frac{J}{m\,s\,K}\right] \qquad \left[\frac{K}{m}\right]$$

or: $$\underset{\text{SPECIFIC HEAT FLOW}}{\vec{j}_u} = - \underset{\text{THERMAL DIFFUSIVITY}}{D} \; \vec{\nabla} u(\vec{r},t) \qquad \text{with} \quad \begin{cases} \vec{j}_u = \dfrac{\vec{j}_e}{c\rho} \\ D = \dfrac{K_0}{c\rho} \end{cases}$$

$$\left[\frac{K\,m}{s}\right] \qquad \left[\frac{m^2}{s}\right] \qquad \left[\frac{K}{m}\right]$$

- Ohm's law of electrical conduction:

$$\underset{\text{CURRENT DENSITY}}{\vec{j}} = \underset{\text{CONDUCTIVITY}}{\sigma} \; \underset{\text{ELECTRIC FIELD}}{\vec{E}} = - \sigma \; \underset{\text{GRADIENT IN ELECTRICAL POTENTIAL}}{\vec{\nabla} V(\vec{r},t)}$$

$$\left[\frac{A}{m^2}\right] \qquad \left[\frac{A}{V\,m}\right] = \left[\frac{1}{\Omega\, m}\right] \qquad \left[\frac{V}{m}\right] \qquad \left[\frac{V}{m}\right]$$

DIVERGENCE:

$$\vec{\nabla} \cdot \vec{F} = \operatorname{div} \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

expresses local source/sink in $\vec{F}$, as outward "FLUX" of $\vec{F}$:

$$\operatorname{div} \vec{F} = \lim_{V \to 0} \frac{1}{V} \oiint_{S(V)} \vec{F} \cdot \vec{n} \, dS$$

Divergence Theorem:

$$\iiint_V \operatorname{div} \vec{F} \, dV = \oiint_{S(V)} \vec{F} \cdot \vec{n} \cdot dS$$

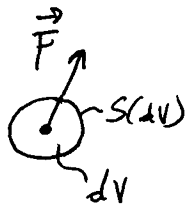
$\downarrow$ SOURCE INSIDE VOLUME

$\downarrow$ FLUX THROUGH SURFACE

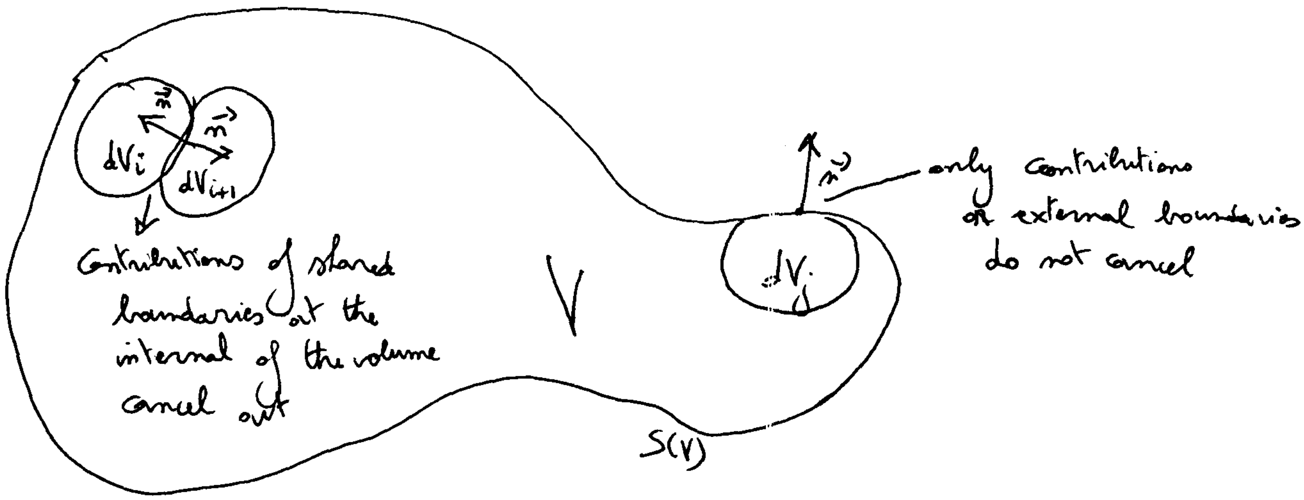
Proof:

$$\oiint_{S(dV)} \vec{F} \cdot \vec{n} \, dS = \operatorname{div} \vec{F} \cdot dV \quad \text{for} \quad dV \to 0$$

(definition of divergence)



$$\sum_i \oiint_{S(dV_i)} \vec{F} \cdot \vec{n} \cdot dS = \sum_i \operatorname{div} \vec{F} \cdot dV_i$$

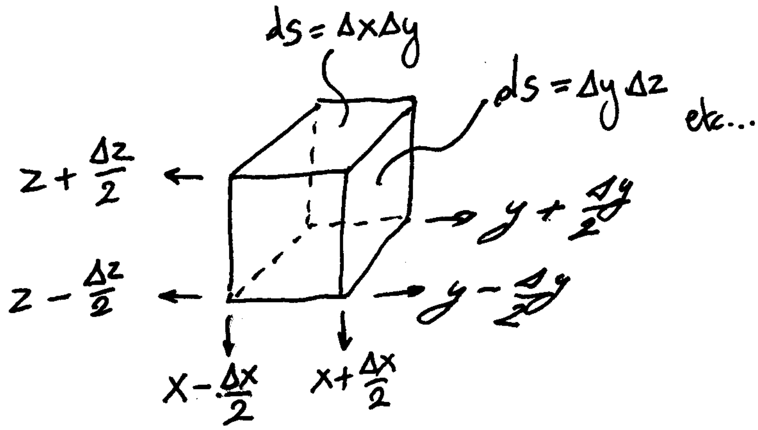


$$\Rightarrow \sum_{i \in \text{outside}} \oiint_{\substack{S(dV_i) \\ \text{outside}}} \vec{F} \cdot \vec{n} \cdot dS = \sum_i \operatorname{div} \vec{F} \cdot dV_i$$

$$\Rightarrow \oiint_{S(V)} \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div} \vec{F} \cdot dV$$

e.g., Cartesian coordinates:

$$\operatorname{div}\vec{F} = \lim_{V\to 0} \frac{1}{V} \oint\!\!\oint_{S(V)} \vec{F}\cdot\vec{n}\,dS$$



$$V = \Delta x . \Delta y . \Delta z$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \frac{1}{\Delta x\Delta y\Delta z}\Big(F_x(x+\tfrac{\Delta x}{2},.,.)\Delta y\Delta z - F_x(x-\tfrac{\Delta x}{2},.,.)\Delta y\Delta z$$

$$+ F_y(.,y+\tfrac{\Delta y}{2},.)\Delta x\Delta z - F_y(.,y-\tfrac{\Delta y}{2},.)\Delta x\Delta z$$

$$+ F_z(.,.,z+\tfrac{\Delta z}{2})\Delta x\Delta y - F_z(.,.,z-\tfrac{\Delta z}{2})\Delta x\Delta y\Big)$$

$$= \lim_{\Delta x\to 0} \frac{F_x(x+\frac{\Delta x}{2},.,.) - F_x(x-\frac{\Delta x}{2},.,.)}{\Delta x} + \dots \quad \text{(same for } y \text{ \& } z\text{)}$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad \text{ok!}$$

Conservation of mass / energy / charge:

$$\underbrace{\frac{d}{dt}\iiint_V u(\vec{r},t)\, dV}_{\substack{\text{CHANGE IN MASS}\\ \text{INSIDE VOLUME } V}} \;(d^3\vec{r}) = -\underbrace{\oiint_{S(V)} \vec{J}(\vec{r},t).\vec{n}\, dS}_{\substack{\text{OUTWARD FLOW OF MASS}\\ \text{THROUGH VOLUME SURFACE}\\ \text{BOUNDARY } S(V)}} + \underbrace{\iiint_V Q(\vec{r},t)\, dV}_{\substack{\text{GENERATION OF}\\ \text{MASS INSIDE VOLUME}\\ V}}$$

$$= \iiint_V \frac{\partial}{\partial t} u(\vec{r},t)\, dV \qquad\qquad = \iiint_V \vec{\nabla}.\vec{J}(\vec{r},t)\, dV$$

$$\Rightarrow \quad \frac{\partial}{\partial t} u(\vec{r},t) = -\vec{\nabla}.\vec{J}(\vec{r},t) + Q(\vec{r},t)$$

$$= D\, \underbrace{\vec{\nabla}.\vec{\nabla}}_{\Delta\,:\,\text{Laplacian}\ \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}}\, u(\vec{r},t) + Q(\vec{r},t)$$

Similarly:

$$c\rho \frac{\partial}{\partial t} u(\vec{r},t) = -\vec{\nabla}.\vec{J}_e(\vec{r},t) + Q(\vec{r},t)$$

$$= K_0\, \Delta\, u(\vec{r},t) + Q(\vec{r},t)$$

and:

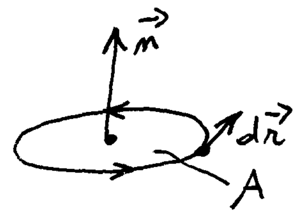
$$\frac{\partial}{\partial t} \underbrace{\rho(\vec{r},t)}_{\substack{\text{CHARGE}\\ \text{DENSITY}}} = -\vec{\nabla}.\vec{J}(\vec{r},t)$$

CURL:

$$\text{curl}\,\vec{F}\ \ (=\text{rot}\,\vec{F}) \;=\; \vec{\nabla}\times\vec{F} \;=\; \begin{pmatrix} \dfrac{\partial F_z}{\partial y}-\dfrac{\partial F_y}{\partial z} \\[2ex] \dfrac{\partial F_x}{\partial z}-\dfrac{\partial F_z}{\partial x} \\[2ex] \dfrac{\partial F_y}{\partial x}-\dfrac{\partial F_x}{\partial y} \end{pmatrix}$$

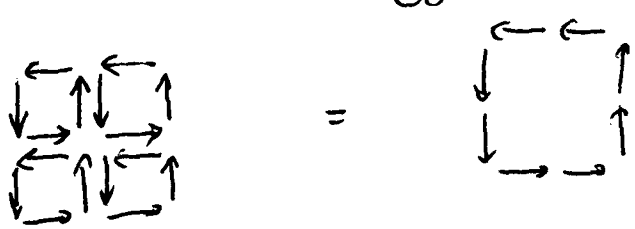
expresses rotational density in field $\vec{F}$:

$$\text{curl}\,\vec{F}\cdot\vec{n} = \lim_{A\to 0}\frac{1}{A}\cdot\oint_{C(A)}\vec{F}\cdot d\vec{r}$$



Stokes Theorem (Kelvin-Stokes Theorem):

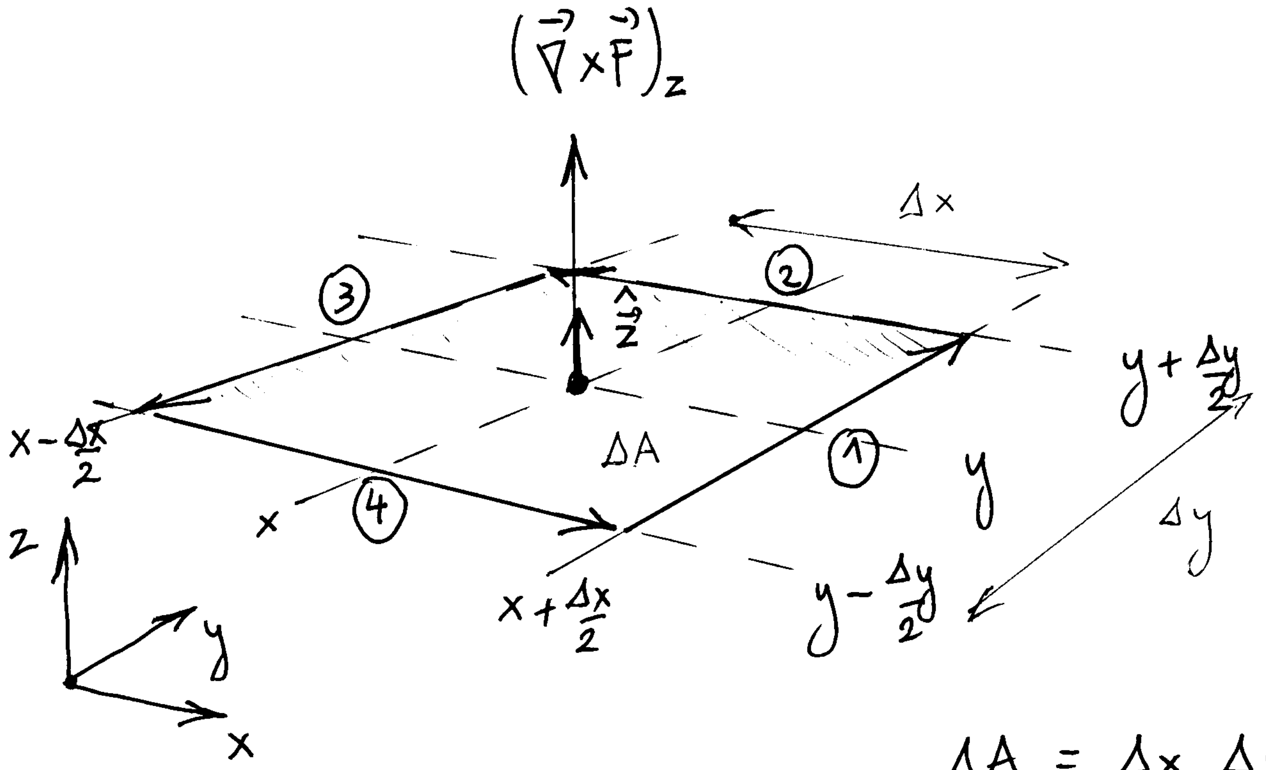
$$\iint_A \text{curl}\,\vec{F}\cdot\vec{n}\,dA = \oint_{C(A)}\vec{F}\cdot d\vec{r}$$



NOTE: $\text{div}\,(\text{curl}\,\vec{F}) = 0$ or $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$

$\text{curl}\,(\text{grad}\,u) = 0$ or $\vec{\nabla}\times(\vec{\nabla}u) = 0$

Example: curl in cartesian coordinates:



$$\Delta A = \Delta x . \Delta y$$

z component of $\vec{\nabla}\times\vec{F}$: (similar for other components by rotation of coordinates)

$$(\vec{\nabla}\times\vec{F})_z = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{1}{\Delta x \Delta y}\Big(\underset{①}{F_y(x+\tfrac{\Delta x}{2}, y, z)}\,\Delta y$$

$$- \underset{②}{F_x(x, y+\tfrac{\Delta y}{2}, z)}\,\Delta x$$

$$- \underset{③}{F_y(x-\tfrac{\Delta x}{2}, y, z)}\,\Delta y$$

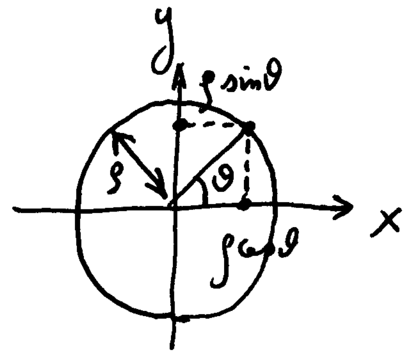
$$+ \underset{④}{F_x(x, y-\tfrac{\Delta y}{2}, z)}\,\Delta x \Big)$$

$$= \lim_{\Delta x\to 0} \frac{F_y(x+\frac{\Delta x}{2}, y, z) - F_y(x-\frac{\Delta x}{2}, y, z)}{\Delta x}$$

$$- \lim_{\Delta y\to 0} \frac{F_x(x, y+\frac{\Delta y}{2}, z) - F_x(x, y-\frac{\Delta y}{2}, z)}{\Delta y} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

VECTOR CALCULUS IN POLAR COORDINATES (2-D)

$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \sin\vartheta \end{cases}$$



$$\vec{\nabla} = \vec{\hat{x}} \frac{\partial}{\partial x} + \vec{\hat{y}} \frac{\partial}{\partial y}$$ Cartesian coordinates

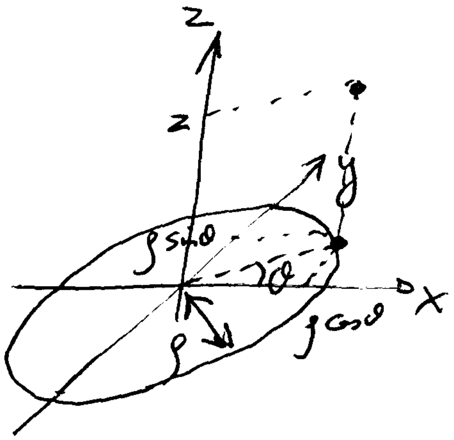
$$= \vec{\hat{\rho}} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \vec{\hat{\theta}} \frac{\partial}{\partial \theta}$$ Polar coordinates

e.g: $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{1}{\rho} \frac{\partial F_\theta}{\partial \theta}$$

VECTOR CALCULUS IN CYLINDRICAL COORDINATES

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \\ z = z \end{cases}$$



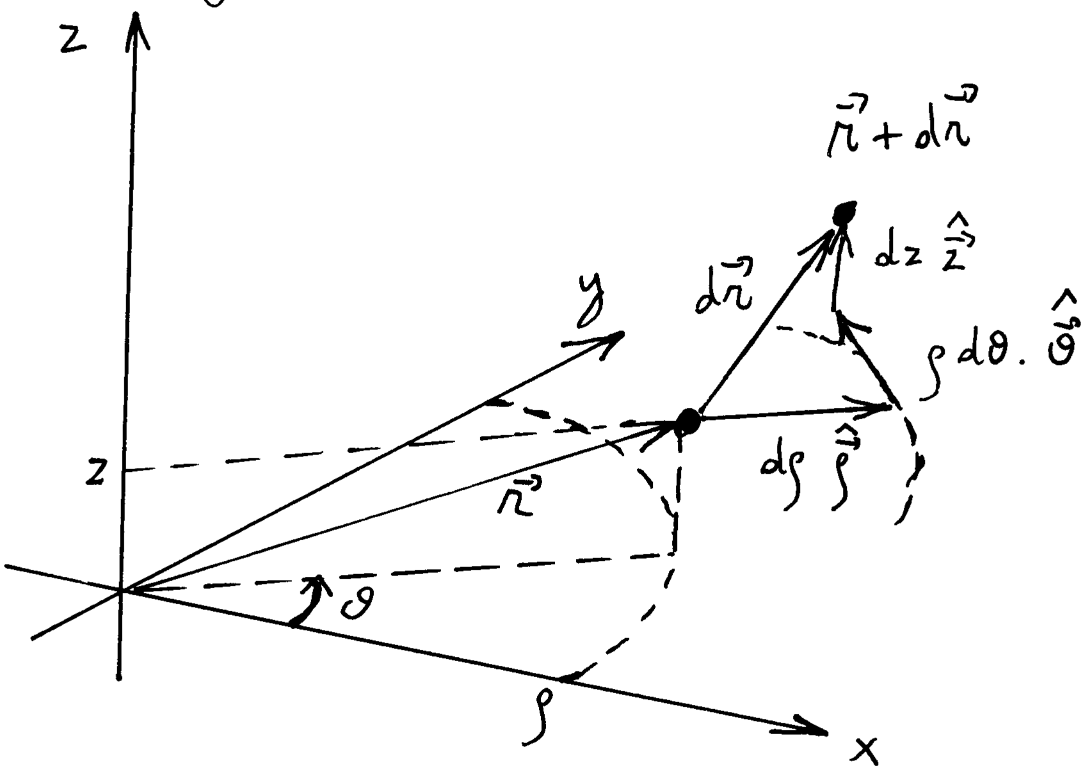
$$\vec{\nabla} = \vec{\hat{x}}\frac{\partial}{\partial x} + \vec{\hat{y}}\frac{\partial}{\partial y} + \vec{\hat{z}}\frac{\partial}{\partial z} \quad \text{Cartesian}$$

$$= \vec{\hat{\rho}}\frac{\partial}{\partial \rho} + \frac{1}{\rho}\vec{\hat{\vartheta}}\frac{\partial}{\partial \vartheta} + \vec{\hat{z}}\frac{\partial}{\partial z} \quad \text{cylindrical}$$

e.g.: $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} =$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\vartheta}{\partial \vartheta} + \frac{\partial F_z}{\partial z}$$

Gradient in cylindrical coordinates:



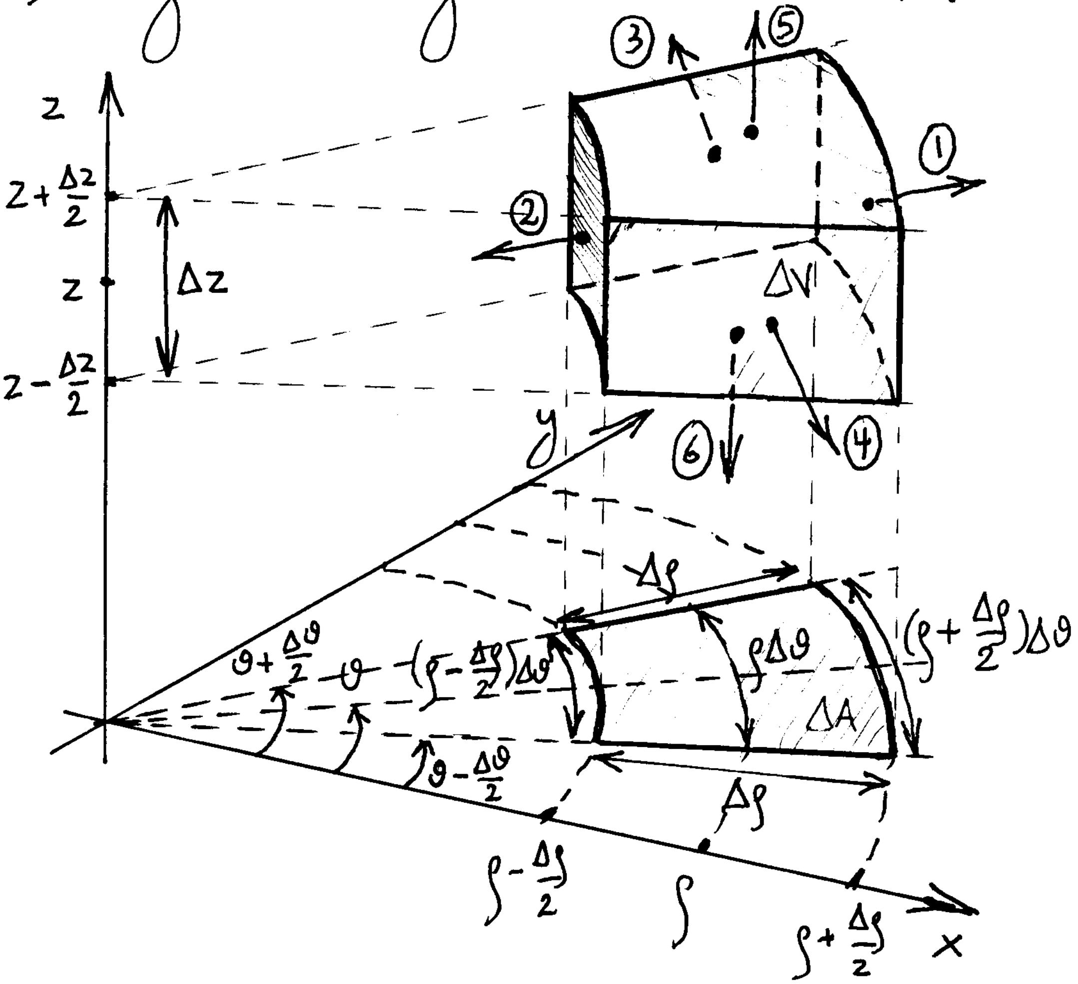
$\vec{F} = \vec{\nabla} u$ or $\vec{F} = F_\rho \hat{\vec{\rho}} + F_\vartheta \hat{\vec{\vartheta}} + F_z \hat{\vec{z}}$

$du = \vec{F} \cdot d\vec{r}$ where $d\vec{r} = d\rho\,\hat{\vec{\rho}} + \rho\, d\vartheta\,\hat{\vec{\vartheta}} + dz\,\hat{\vec{z}}$

$$\frac{\partial u}{\partial \rho} d\rho + \frac{\partial u}{\partial \vartheta} d\vartheta + \frac{\partial u}{\partial z} dz = F_\rho\, d\rho + F_\vartheta\, \rho\, d\vartheta + F_z\, dz$$

$$\Rightarrow \begin{cases} F_\rho = \dfrac{\partial u}{\partial \rho} \\ F_\vartheta = \dfrac{1}{\rho} \dfrac{\partial u}{\partial \vartheta} \\ F_z = \dfrac{\partial u}{\partial z} \end{cases}$$

Divergence in cylindrical coordinates:



$$\Delta V = \Delta\varrho\, \varrho\Delta\vartheta\, \Delta z$$

$$\Delta A = \Delta\varrho\, \varrho\Delta\vartheta$$

$$\vec{\nabla}\cdot\vec{F} = \lim_{\substack{\Delta\varrho\to 0\\ \Delta\vartheta\to 0\\ \Delta z\to 0}} \frac{1}{\Delta\varrho\,\varrho\Delta\vartheta\,\Delta z}\cdot\Big(\underset{1}{F_\varrho(\varrho+\tfrac{\Delta\varrho}{2},\vartheta,z)\cdot(\varrho+\tfrac{\Delta\varrho}{2})\Delta\vartheta\Delta z} - \underset{2}{F_\varrho(\varrho-\tfrac{\Delta\varrho}{2},\vartheta,z)\cdot(\varrho-\tfrac{\Delta\varrho}{2})\Delta\vartheta\Delta z}$$

$$+ \underset{3}{F_\vartheta(\varrho,\vartheta+\tfrac{\Delta\vartheta}{2},z)\Delta\varrho\Delta z} - \underset{4}{F_\vartheta(\varrho,\vartheta-\tfrac{\Delta\vartheta}{2},z)\Delta\varrho\Delta z}$$

$$+ \underset{5}{F_z(\varrho,\vartheta,z+\tfrac{\Delta z}{2})\varrho\Delta\vartheta\Delta z} - \underset{6}{F_z(\varrho,\vartheta,z-\tfrac{\Delta z}{2})\varrho\Delta\vartheta\Delta z}\Big)$$

$$= \frac{1}{\varrho}\lim_{\Delta\varrho\to 0} \frac{F_\varrho(\varrho+\frac{\Delta\varrho}{2},\vartheta,z)\cdot(\varrho+\frac{\Delta\varrho}{2}) - F_\varrho(\varrho-\frac{\Delta\varrho}{2},\vartheta,z)\cdot(\varrho-\frac{\Delta\varrho}{2})}{\Delta\varrho}$$

$$+ \frac{1}{\varrho}\lim_{\Delta\vartheta\to 0} \frac{F_\vartheta(\varrho,\vartheta+\frac{\Delta\vartheta}{2},z) - F_\vartheta(\varrho,\vartheta-\frac{\Delta\vartheta}{2},z)}{\Delta\vartheta}$$

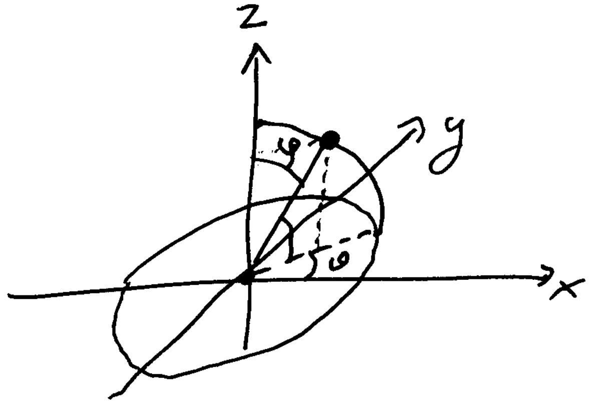
$$+ \lim_{\Delta z\to 0} \frac{F_z(\varrho,\vartheta,z+\frac{\Delta z}{2}) - F_z(\varrho,\vartheta,z-\frac{\Delta z}{2})}{\Delta z} = \frac{1}{\varrho}\frac{\partial}{\partial\varrho}(\varrho F_\varrho) + \frac{1}{\varrho}\frac{\partial F_\vartheta}{\partial\vartheta} + \frac{\partial F_z}{\partial z}$$

VECTOR CALCULUS IN SPHERICAL COORDINATES

$$x = r\cos\vartheta \cdot \sin\varphi$$

$$y = r\sin\vartheta \cdot \sin\varphi$$

$$z = r\cos\varphi$$



$$\vec{\nabla} = \vec{\hat{r}}\,\frac{\partial}{\partial r} + \frac{1}{r}\,\vec{\hat{\varphi}} \cdot \frac{\partial}{\partial \varphi} + \frac{1}{r\sin\varphi}\,\vec{\hat{\vartheta}} \cdot \frac{\partial}{\partial \vartheta}$$

e.g. $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} =$

$$\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\varphi}\,\frac{\partial}{\partial\varphi}\left(\sin\varphi\, F_\varphi\right) + \frac{1}{r\sin\varphi} \cdot \frac{\partial F_\vartheta}{\partial\vartheta}$$

LAPLACIAN:

$$\Delta u = \vec{\nabla}^2 u = \vec{\nabla} \cdot \vec{\nabla} u = \operatorname{div}(\operatorname{grad} u)$$

CARTESIAN:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

CYLINDRICAL (POLAR):

$$\Delta u = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

SPHERICAL:

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\varphi} \frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi} \right)$$

$$+ \frac{1}{r^2 \sin^2\varphi} \frac{\partial^2 u}{\partial \theta^2}$$