Lecture 14

Diffusion in Polar and Cylindrical Coordinates

References

Haberman APDE, Sec. 7.7, 7.8 and 7.9.

http://en.wikipedia.org/wiki/Cylindrical_coordinate_system

http://en.wikipedia.org/wiki/Bessel_function

http://en.wikipedia.org/wiki/Fourier–Bessel_series

LAPLACIAN:

$$\Delta u = \vec{\nabla}^2 u = \vec{\nabla} \cdot \vec{\nabla} u = \mathrm{div}(\mathrm{grad}\, u)$$

CARTESIAN:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

CYLINDRICAL (POLAR):

$$\Delta u = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

SPHERICAL:

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi} \frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi}\right)$$

$$+ \frac{1}{r^2 \sin^2\varphi} \frac{\partial^2 u}{\partial \theta^2}$$

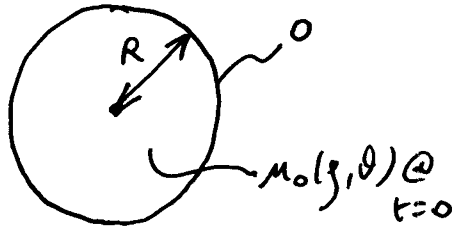
e.g.: Diffusion in polar coordinates:

$$D \nabla^2 u = \frac{\partial u}{\partial t}, \quad \text{or} \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{D}\frac{\partial u}{\partial t}$$

I.C: $u(\rho, \theta, 0) = u_0(\rho, \theta)$

B.C: $u(R, \theta, t) = 0$ (VALUE)

$u(0, \theta, t) =$ continuous/finite ("FLUX")



Separation of variables: $u(\rho, \theta, t) = f(\rho) \cdot g(\theta) \cdot h(t)$

B.C: $f(R) = 0$

$\frac{df}{d\rho}(0) = 0$ (ODD symmetry) or $f(0) = 0$ (EVEN symmetry)

$g(\theta + 2\pi) = g(\theta)$

$$\Rightarrow \underbrace{\frac{\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{df}{d\rho}\right)}{f} + \frac{\frac{1}{\rho^2}\frac{d^2 g}{d\theta^2}}{g}}_{\text{function of } \rho \,\&\, \theta \text{ only}} = \underbrace{\frac{\frac{1}{D}\frac{dh}{dt}}{h}}_{\text{function of } t \text{ only}} = \underset{\text{constant}}{-\lambda}$$

$$\Rightarrow \frac{dh}{dt} = -\lambda D h, \quad \text{or} \quad h = C e^{-\lambda D t}$$

Separation continued (multiplying by ρ^2):

$$\frac{\rho \frac{d}{d\rho}\left(\rho \frac{df}{d\rho}\right)}{f} + \lambda \rho^2 = -\frac{\frac{d^2 g}{d\theta^2}}{g} = n^2$$

constant

function of ρ only — function of θ only

$$\Rightarrow \quad \frac{d^2 g}{d\theta^2} + n^2 g = 0, \quad \text{or} \quad g = A \cos n\theta + B \sin n\theta$$

where n = integer to satisfy B.C.

$$\Rightarrow \quad \rho \frac{d}{d\rho}\left(\rho \frac{df}{d\rho}\right) + (\lambda \rho^2 - n^2) f = 0$$

$$\rho \frac{d}{d\rho}\left(\rho \frac{df}{d\rho}\right) = \rho^2 \frac{d^2 f}{d\rho^2} + \rho \frac{df}{d\rho}$$

Solutions are BESSEL FUNCTIONS

Let $\rho = \frac{x}{\sqrt{\lambda}}$, or $\lambda \rho^2 = x^2$

$$\rho \frac{df}{d\rho} = x \frac{df}{dx}$$

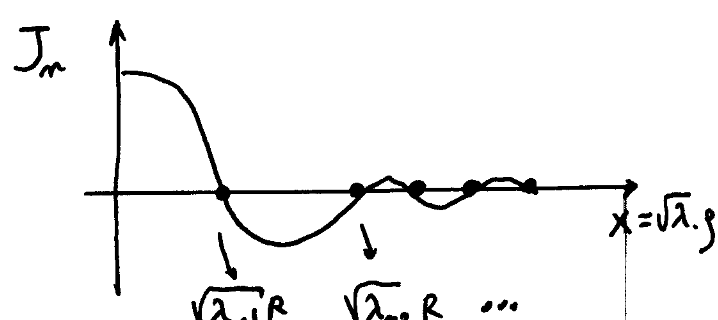
$$\rho^2 \frac{d^2 f}{d\rho^2} = x^2 \frac{d^2 f}{dx^2}$$

$$\Rightarrow x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - n^2) f = 0$$

or $f(x) = J_n(x)$ Bessel function $n = 0, 1, 2, \ldots$

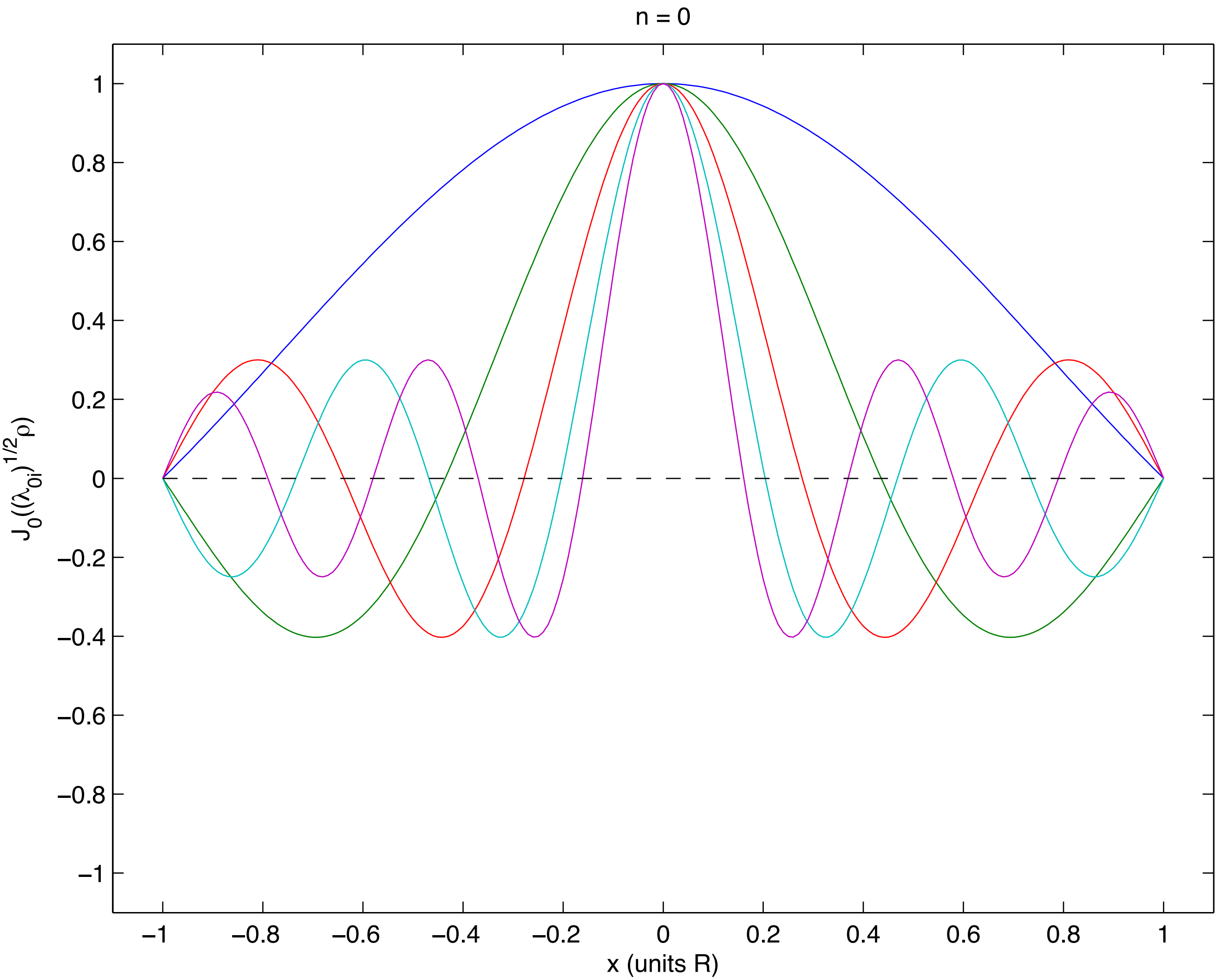
$$\Rightarrow f(\rho) = J_n(\sqrt{\lambda}\,\rho)$$

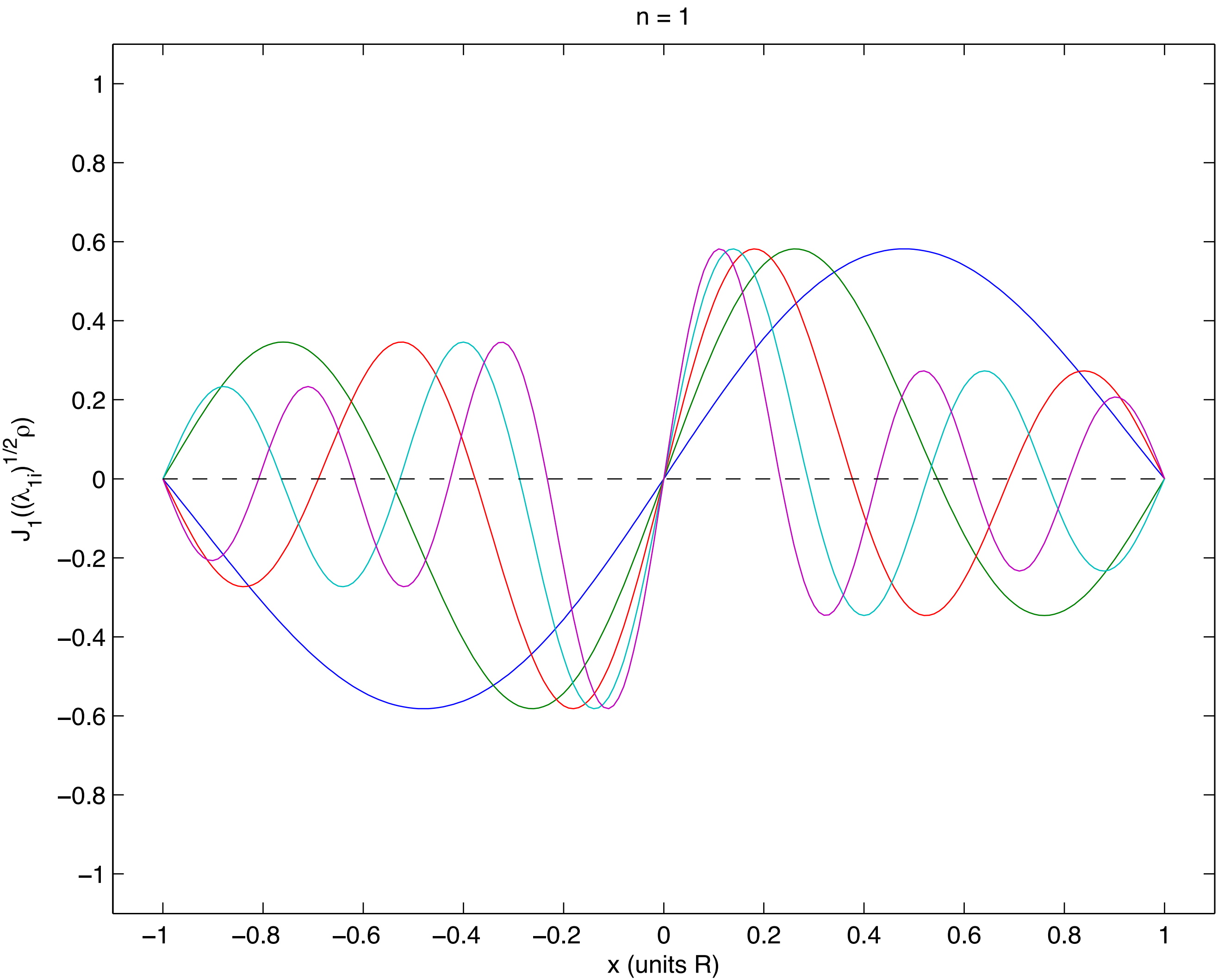
B.C $f(R) = 0 \Rightarrow \sqrt{\lambda_{ni}}\, R$ = roots of J_n



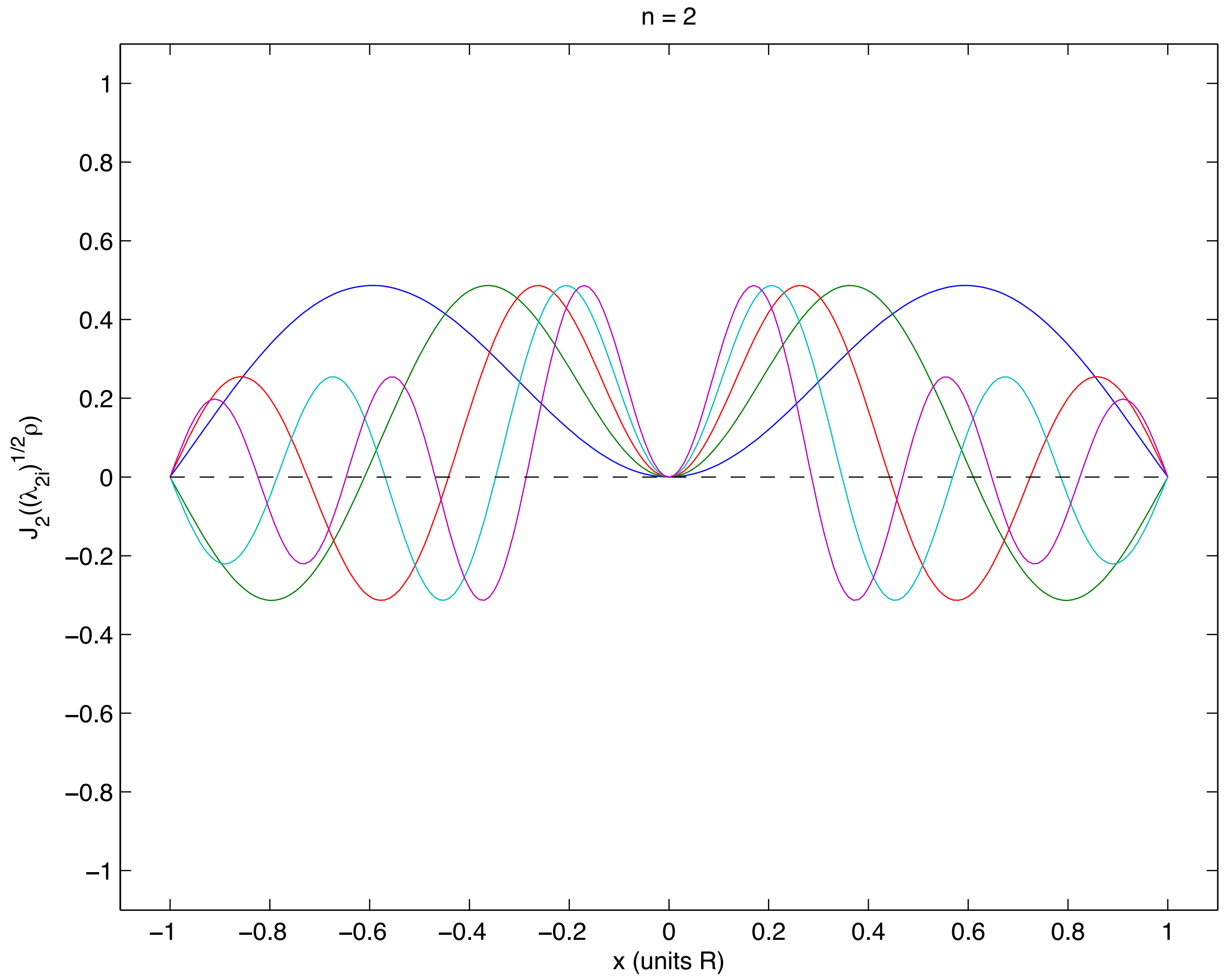
$$\Rightarrow u(\rho, \theta, t) = \sum_n \sum_i J_n(\sqrt{\lambda_{ni}}\,\rho)\left(A_{ni}\cos n\theta + B_{ni}\sin n\theta\right) e^{-\lambda_{ni} D t}$$

where A_{ni} and B_{ni} are given by I.C.

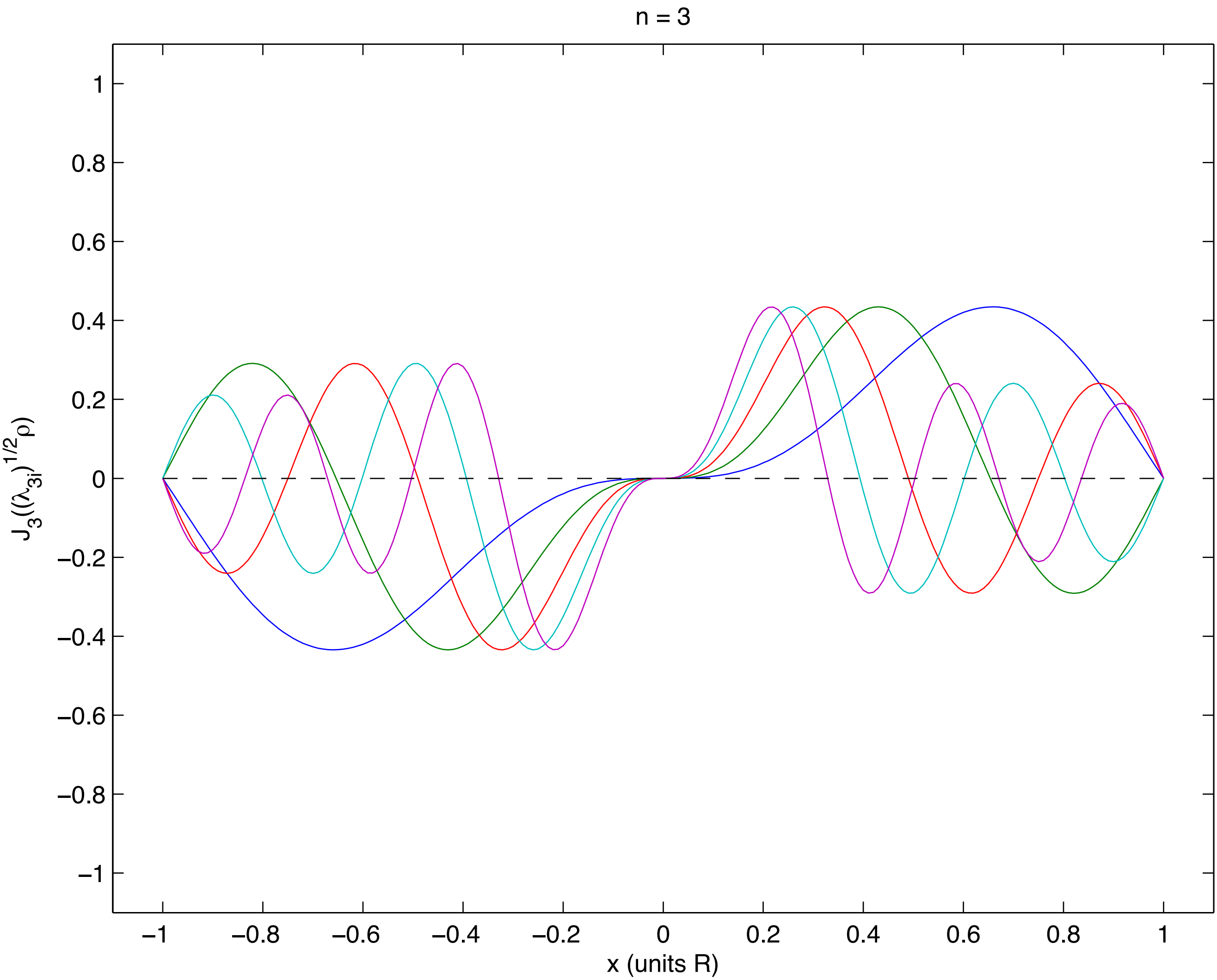

n = 0
J_0((λ_0i)^1/2 ρ)
x (units R)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



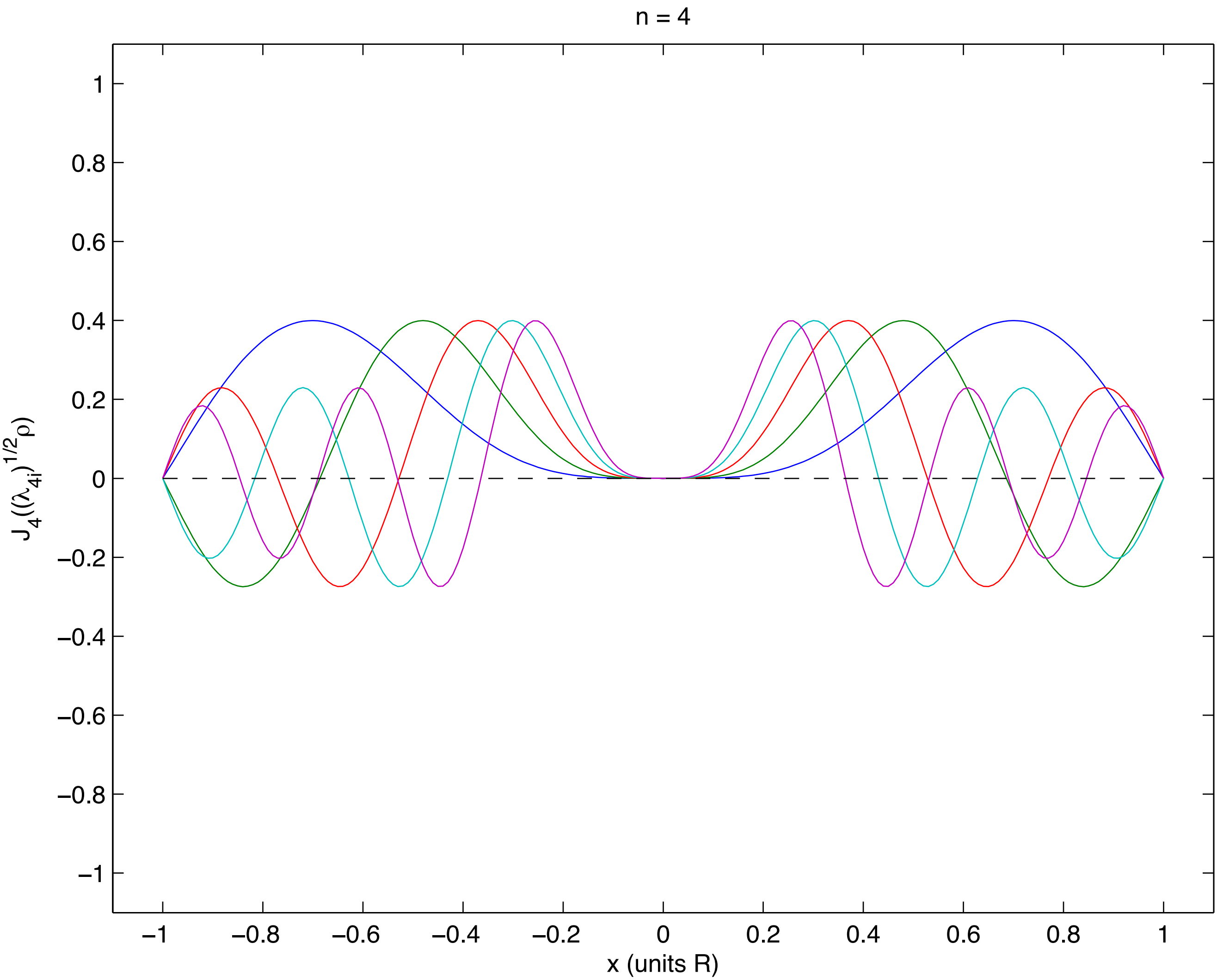
n = 1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x (units R)
J1((λ1i)1/2ρ)



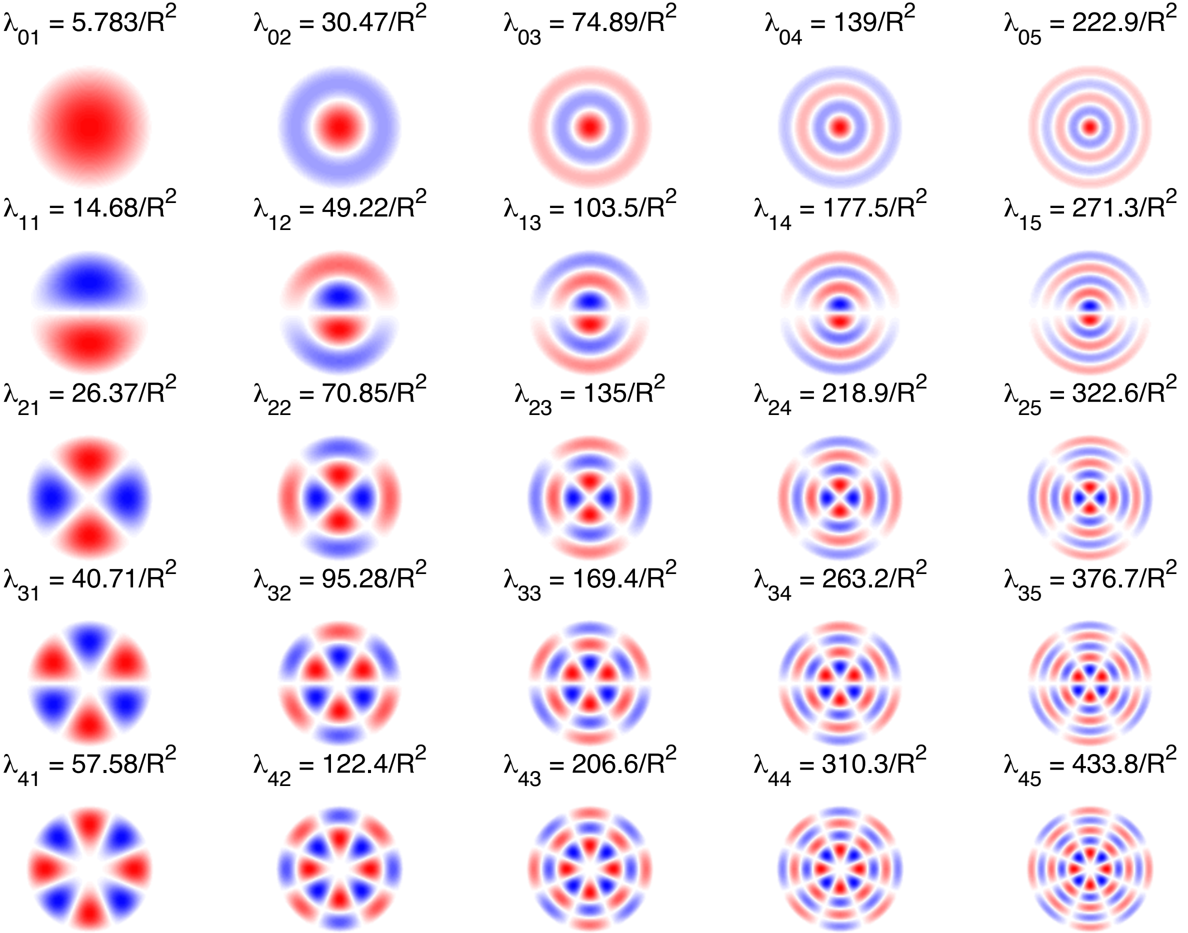
n = 2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
$J_2((\lambda_{2i})^{1/2}\rho)$
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x (units R)

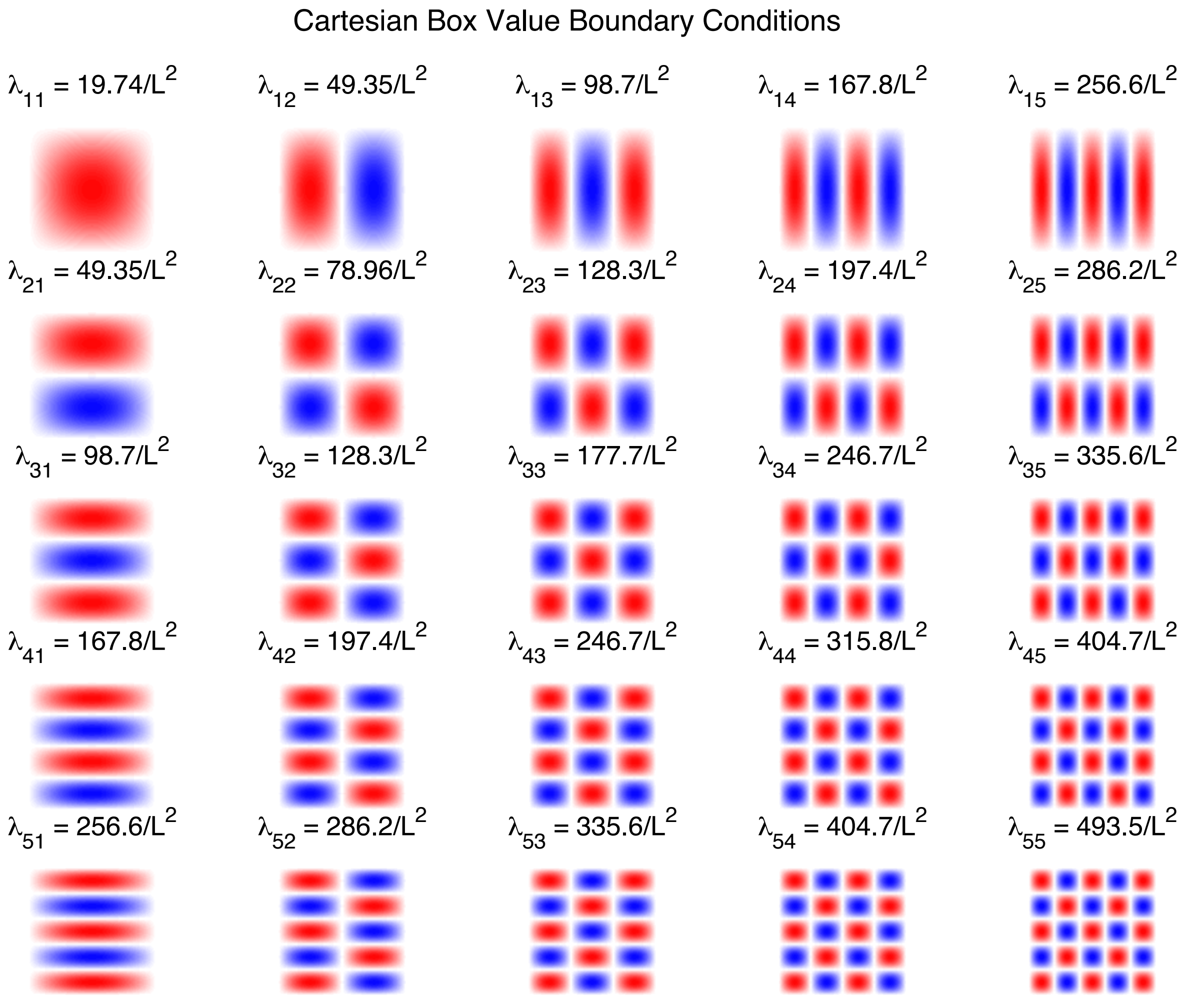


n = 3
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
$J_3((\lambda_{3i})^{1/2}\rho)$
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x (units R)


n = 4
$J_4((\lambda_{4i})^{1/2}\rho)$
x (units R)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Cylindrical Value Boundary Conditions



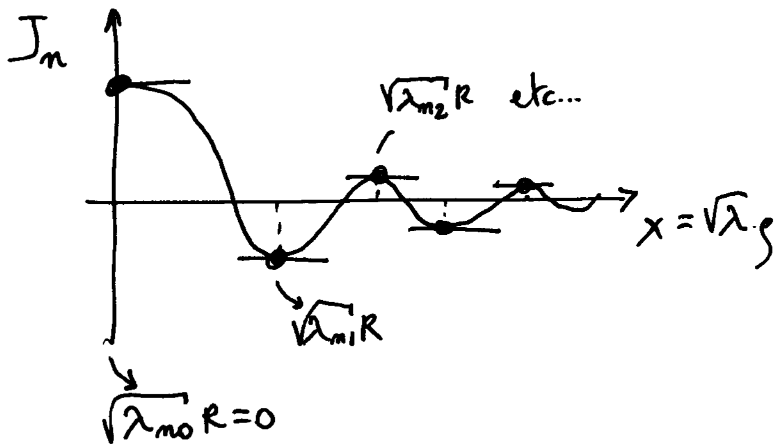


Cartesian Box Value Boundary Conditions
$\lambda_{11} = 19.74/L^2$
$\lambda_{12} = 49.35/L^2$
$\lambda_{13} = 98.7/L^2$
$\lambda_{14} = 167.8/L^2$
$\lambda_{15} = 256.6/L^2$
$\lambda_{21} = 49.35/L^2$
$\lambda_{22} = 78.96/L^2$
$\lambda_{23} = 128.3/L^2$
$\lambda_{24} = 197.4/L^2$
$\lambda_{25} = 286.2/L^2$
$\lambda_{31} = 98.7/L^2$
$\lambda_{32} = 128.3/L^2$
$\lambda_{33} = 177.7/L^2$
$\lambda_{34} = 246.7/L^2$
$\lambda_{35} = 335.6/L^2$
$\lambda_{41} = 167.8/L^2$
$\lambda_{42} = 197.4/L^2$
$\lambda_{43} = 246.7/L^2$
$\lambda_{44} = 315.8/L^2$
$\lambda_{45} = 404.7/L^2$
$\lambda_{51} = 256.6/L^2$
$\lambda_{52} = 286.2/L^2$
$\lambda_{53} = 335.6/L^2$
$\lambda_{54} = 404.7/L^2$
$\lambda_{55} = 493.5/L^2$

NOTES:

- Flux / Value B.C. @ $\rho = 0$ is automatically taken care of.
- What if the B.C @ $\rho = R$ is of the FLUX type?

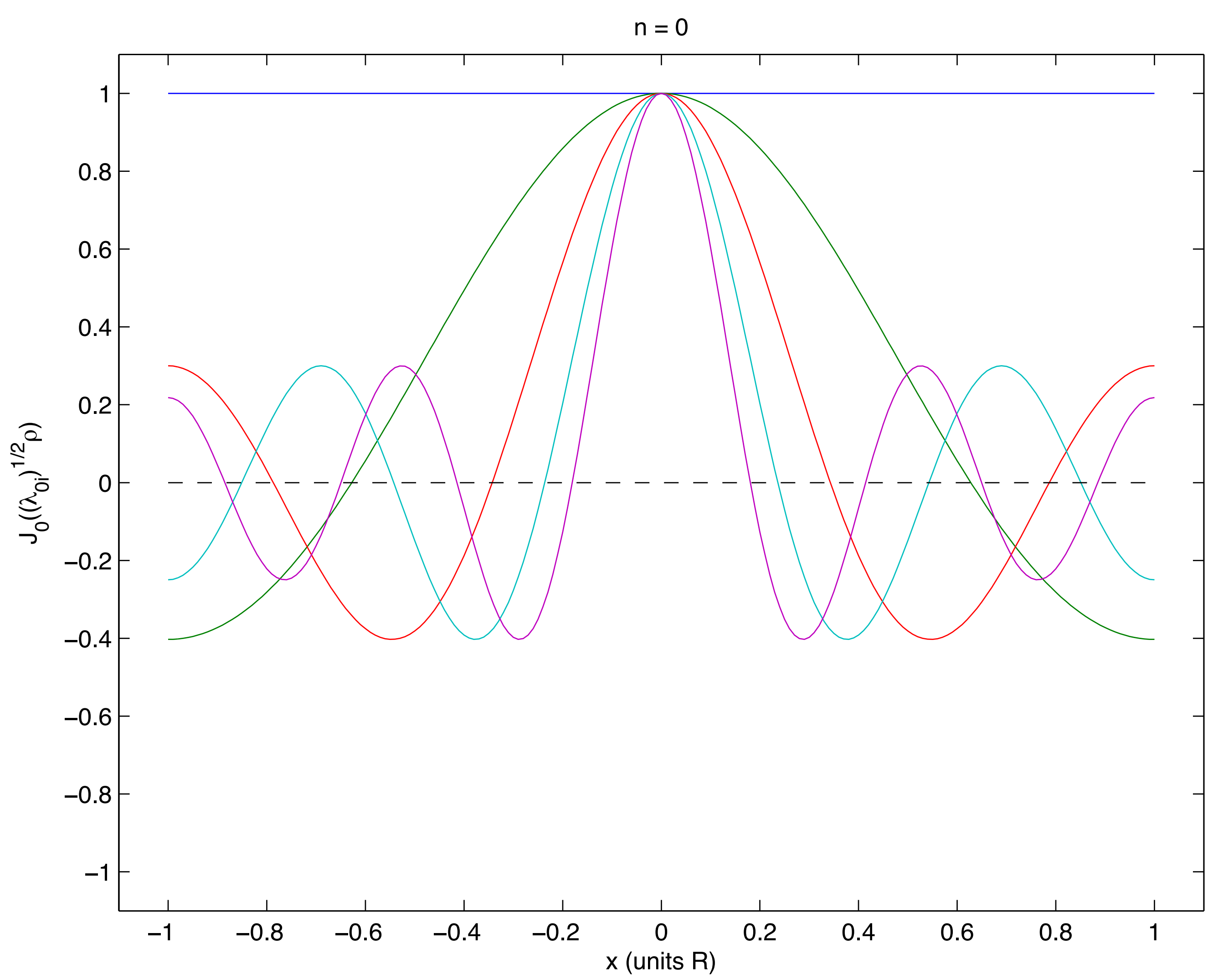
$$\Rightarrow \quad \frac{df}{d\rho}(R) = 0, \quad \text{or} \quad \sqrt{\lambda_{ni}}R = \text{extrema of } J_n$$
$$= \text{roots of } \frac{dJ_n}{dx}$$



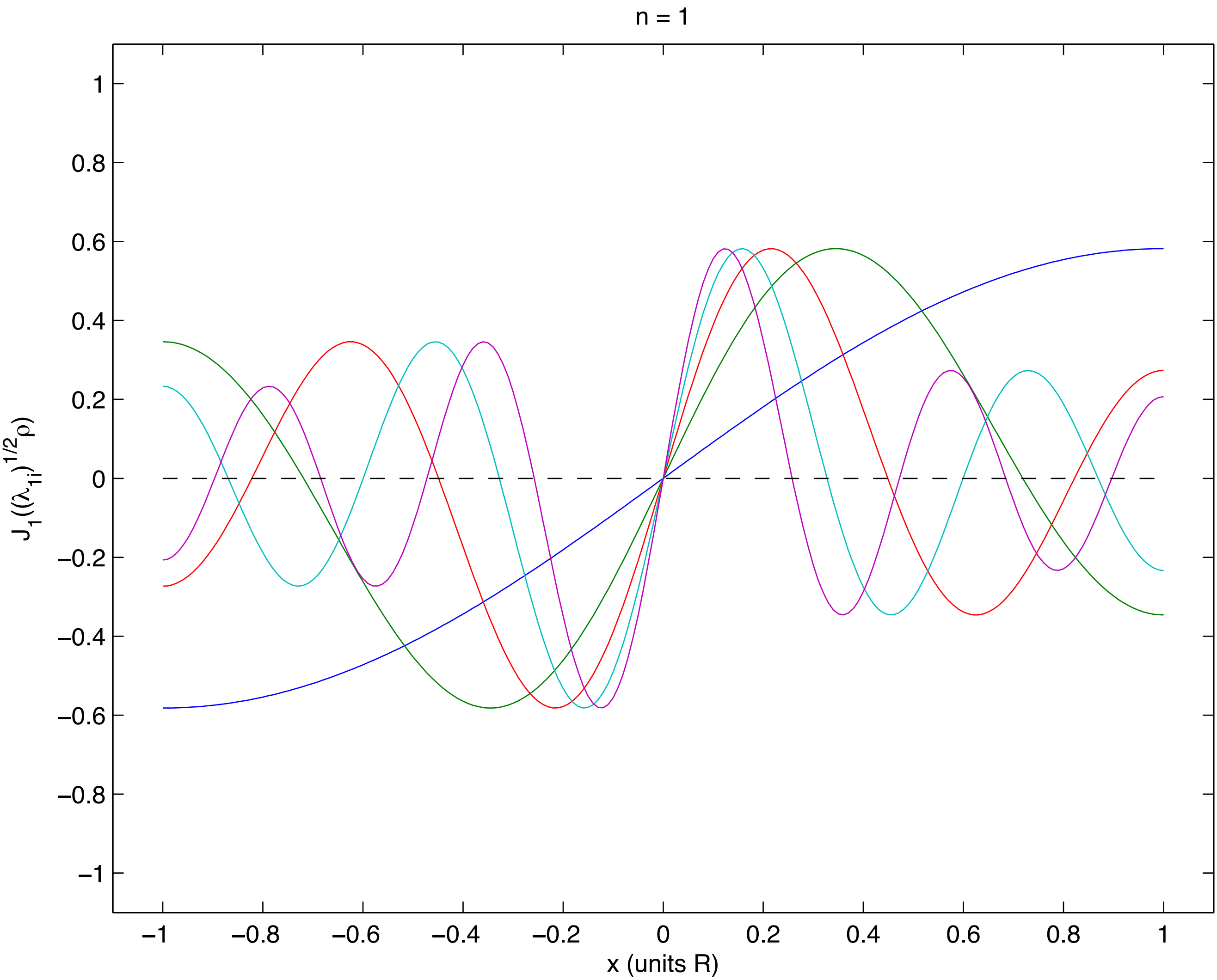
$\lambda_{00} = 0$! Also $J_n(0) = 0$ for $n = 1, 2, \ldots$
$J_0(0) = 1$

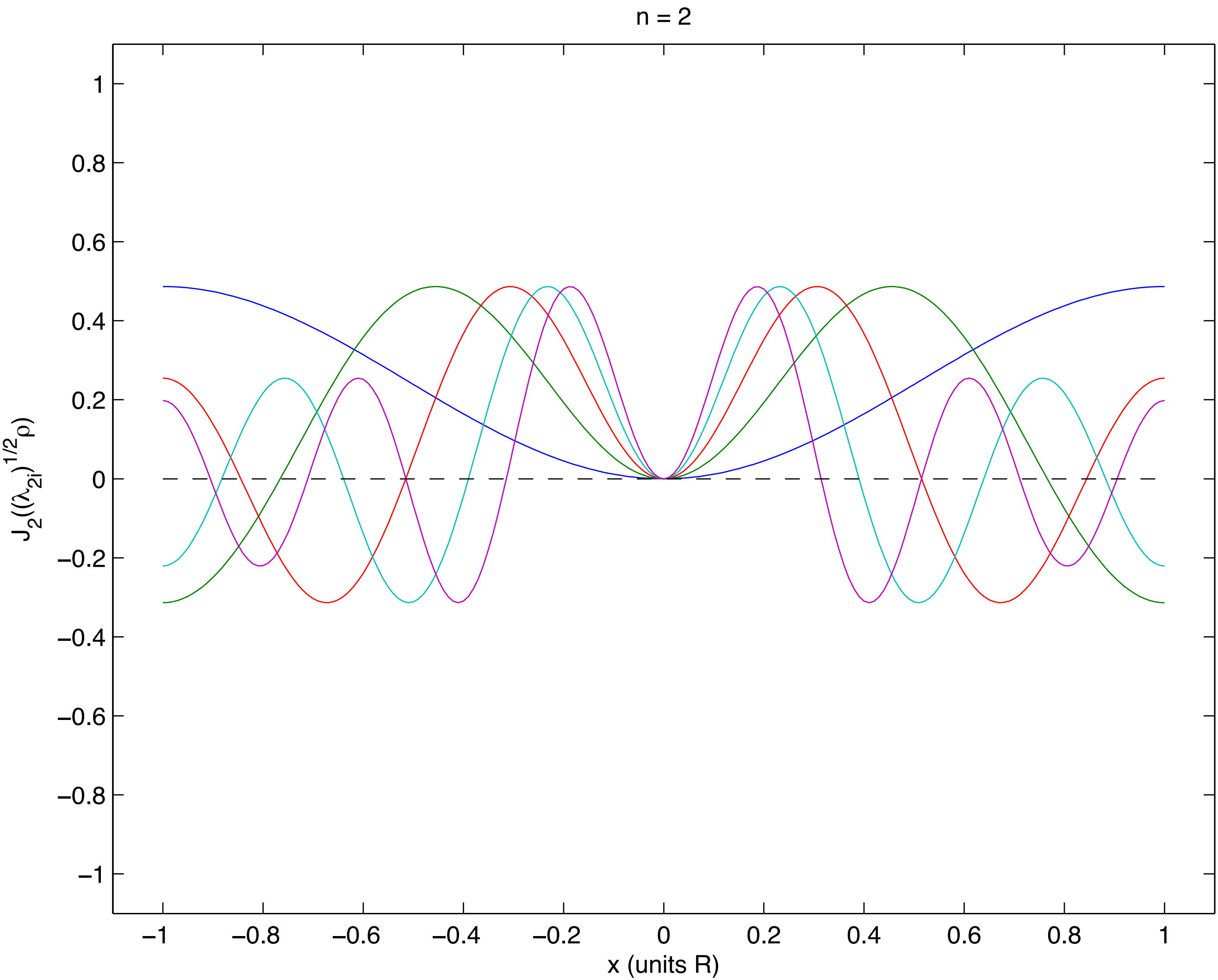
$$\Rightarrow u(\rho, \theta, t) = A_{00} + \sum_n \sum_i J_n(\sqrt{\lambda_{ni}}\rho)(A_{ni}\cos n\theta + B_{ni}\sin n\theta)e^{-\lambda_{ni}Dt}$$

A_{00} → constant solution @ $t \to \infty$

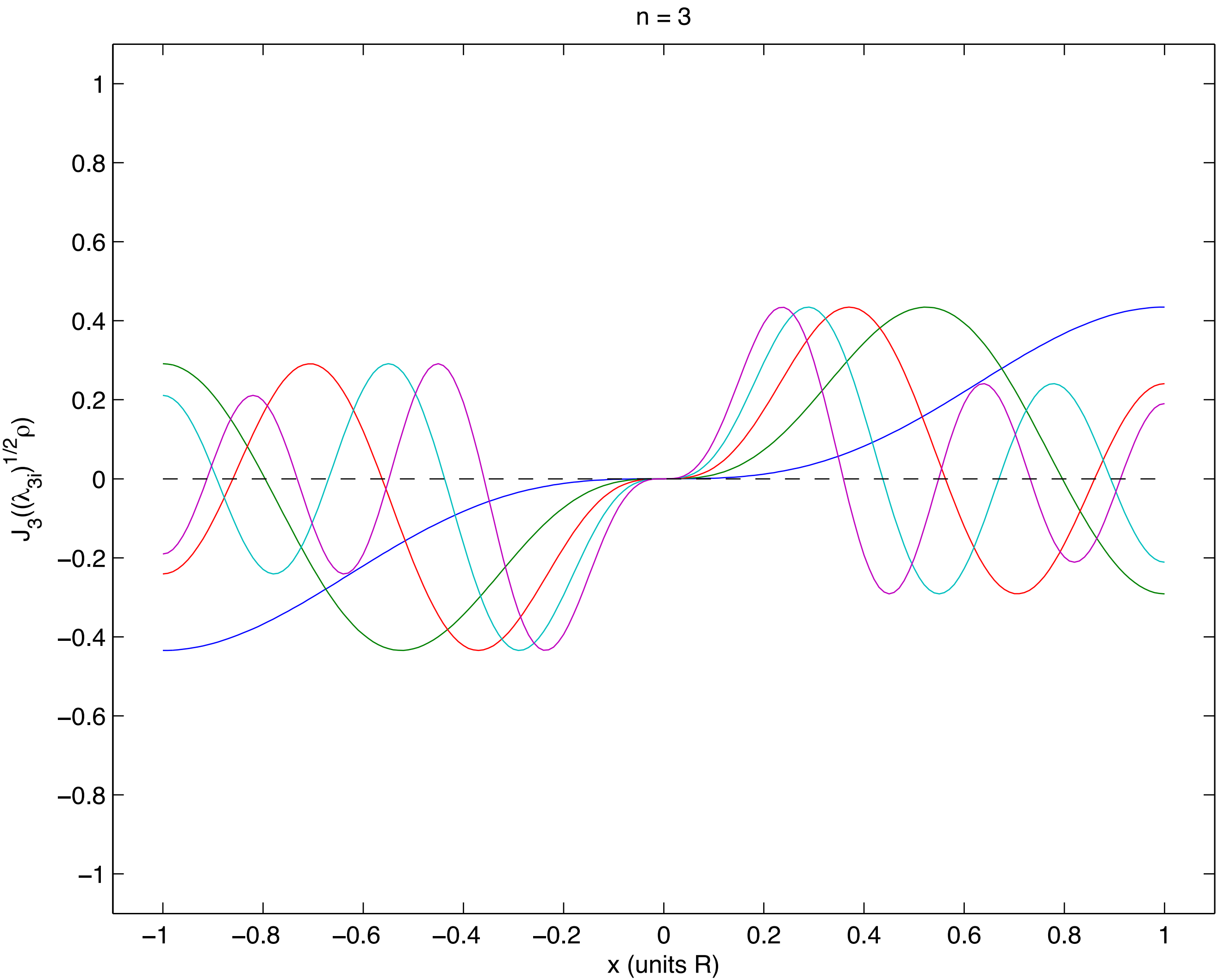


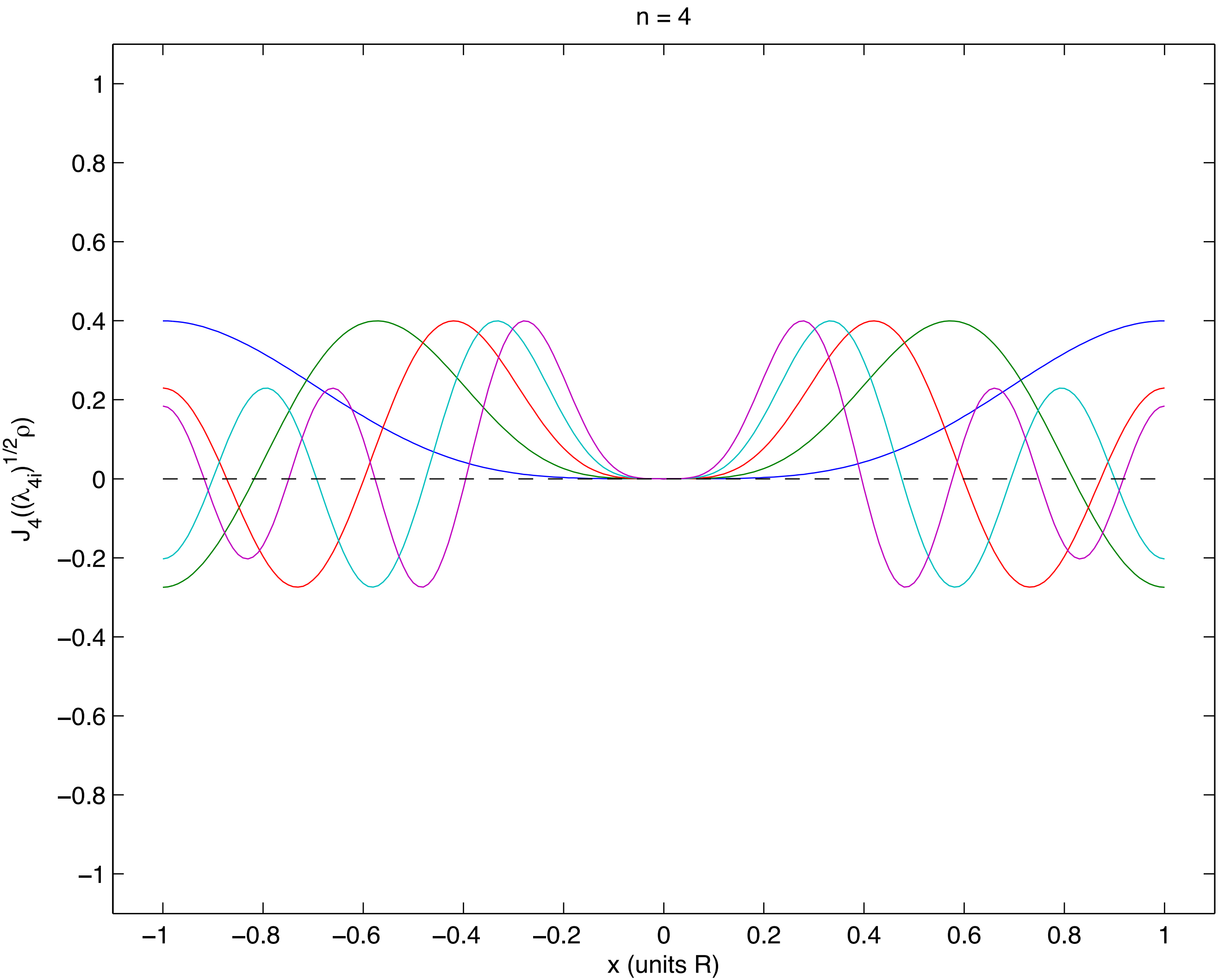
n = 0
J0((λ0i)1/2ρ)
x (units R)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


n = 1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
$J_1((\lambda_{1i})^{1/2}\rho)$
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x (units R)

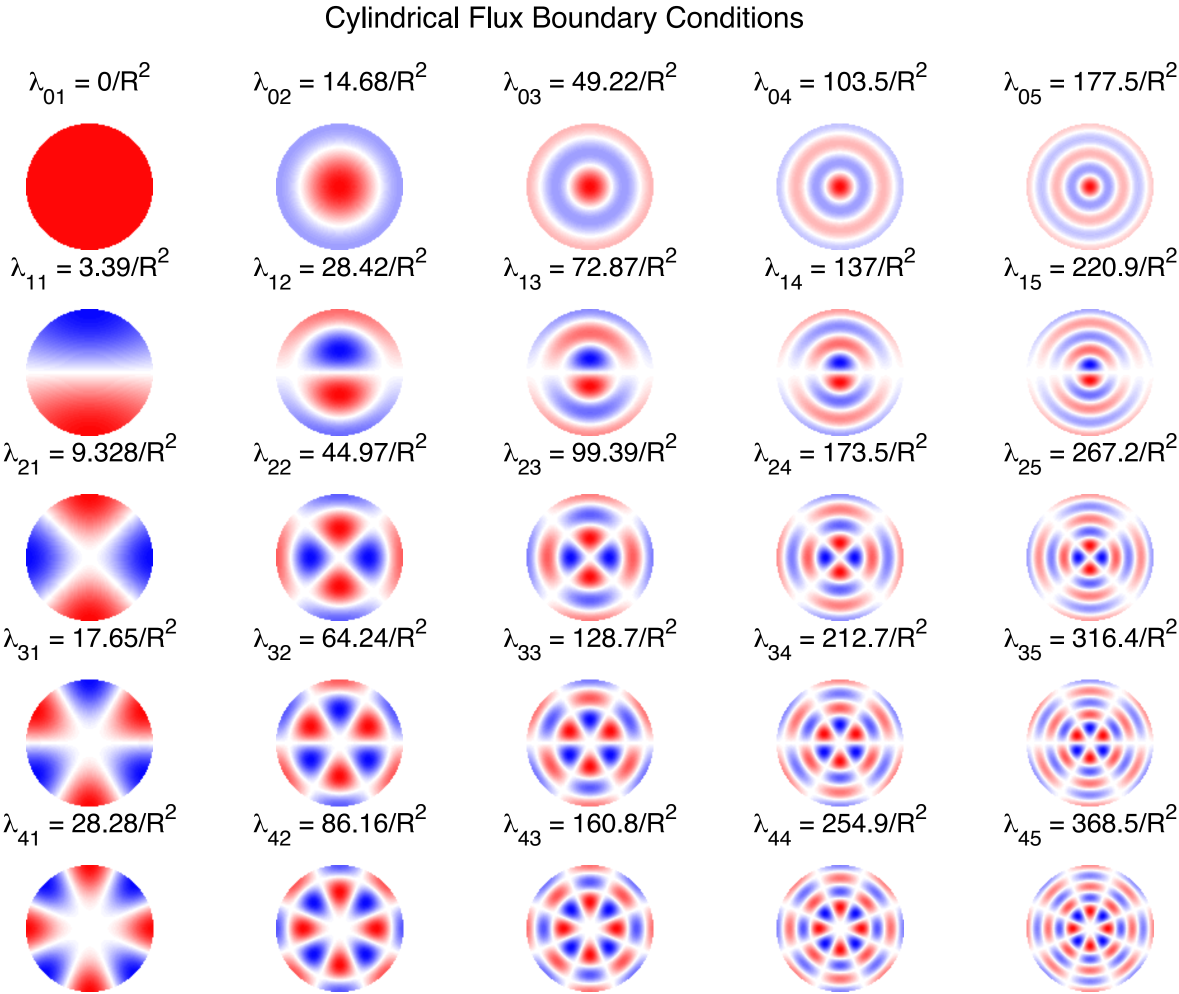


n = 2
$J_2((\lambda_{2i})^{1/2}\rho)$
x (units R)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

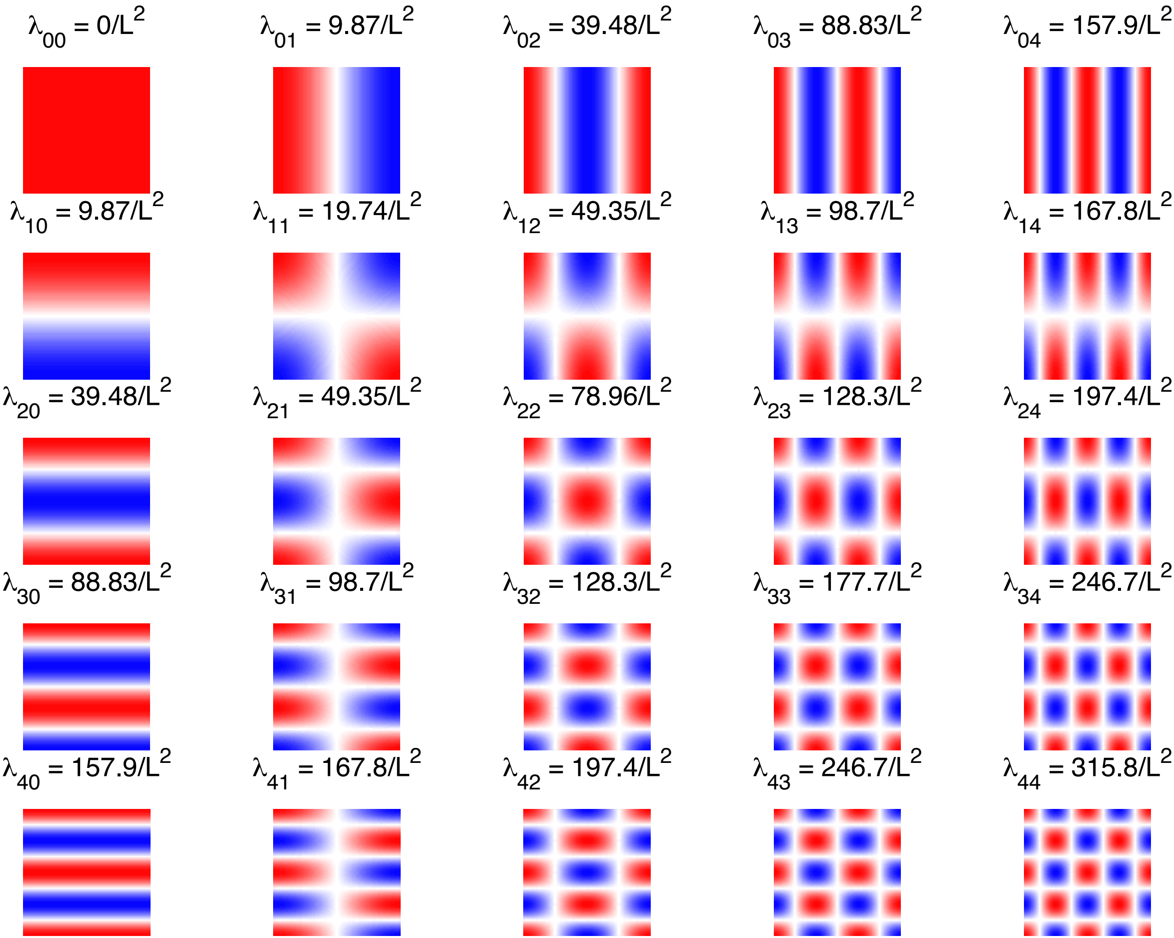

n = 3
$J_3((\lambda_{3i})^{1/2}\rho)$
x (units R)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



n = 4
$J_4((\lambda_{4i})^{1/2}\rho)$
x (units R)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Cylindrical Flux Boundary Conditions
λ01 = 0/R2
λ02 = 14.68/R2
λ03 = 49.22/R2
λ04 = 103.5/R2
λ05 = 177.5/R2
λ11 = 3.39/R2
λ12 = 28.42/R2
λ13 = 72.87/R2
λ14 = 137/R2
λ15 = 220.9/R2
λ21 = 9.328/R2
λ22 = 44.97/R2
λ23 = 99.39/R2
λ24 = 173.5/R2
λ25 = 267.2/R2
λ31 = 17.65/R2
λ32 = 64.24/R2
λ33 = 128.7/R2
λ34 = 212.7/R2
λ35 = 316.4/R2
λ41 = 28.28/R2
λ42 = 86.16/R2
λ43 = 160.8/R2
λ44 = 254.9/R2
λ45 = 368.5/R2

Cartesian Box Flux Boundary Conditions



FOURIER – BESSEL series expansion of I.C. :

$$u_0(\rho,\vartheta) = u(\rho,\vartheta,0) = \sum_n \sum_i J_n(\sqrt{\lambda_{ni}}\,\rho)\left(A_{ni}\cos n\vartheta + B_{ni}\sin n\vartheta\right)$$

where $J_n(\sqrt{\lambda_{ni}}\,\rho)\begin{Bmatrix}\cos n\vartheta\\ \sin n\vartheta\end{Bmatrix}$ and $J_m(\sqrt{\lambda_{mj}}\,\rho)\begin{Bmatrix}\cos m\vartheta\\ \sin m\vartheta\end{Bmatrix}$ are orthogonal over cylinder cross-section area.

$$\Rightarrow \int_0^R\int_0^{2\pi} u_0(\rho_0,\vartheta_0)\, J_n(\sqrt{\lambda_{ni}}\,\rho_0)\begin{Bmatrix}\cos n\vartheta_0\\ \sin n\vartheta_0\end{Bmatrix} \underbrace{\rho_0\, d\rho_0\, d\vartheta_0}_{dA_0 \text{ area in cylinder}}$$

$$= \sum_m \sum_j \underbrace{\int_0^R J_n(\sqrt{\lambda_{ni}}\,\rho_0) J_m(\sqrt{\lambda_{mj}}\,\rho_0)\,\rho_0\, d\rho_0}_{= \delta_{ij}\delta_{mn} \cdot \underbrace{\int_0^R J_n^2(\sqrt{\lambda_{ni}}\,\rho_0)\rho_0\, d\rho_0}_{= \frac{R^2}{2}\cdot J_{n+1}^2(\sqrt{\lambda_{ni}}\,R)}} \cdot \underbrace{\int_0^{2\pi}\begin{Bmatrix}\cos n\vartheta_0\\ \sin n\vartheta_0\end{Bmatrix}\left(A_{mj}\cos m\vartheta_0 + B_{mj}\sin m\vartheta_0\right) d\vartheta_0}_{\substack{= \delta_{ij}\delta_{mn}\begin{Bmatrix}A_{ni}\\ B_{ni}\end{Bmatrix}\cdot\pi \text{ for } n\geq 1 \\ = \delta_{ij}\delta_{mn}\begin{Bmatrix}A_{ni}\\ 0\end{Bmatrix}\cdot 2\pi \text{ for } n=0}}$$

$$\Rightarrow \quad A_{0i} = \frac{\int_0^R \int_0^{2\pi} u_0(\rho_0, \vartheta_0)\, J_0\left(\sqrt{\lambda_{0i}}\, \rho_0\right) \rho_0 \, d\rho_0 \, d\vartheta_0}{\pi R^2 \, J_1^2\left(\sqrt{\lambda_{0i}}\, R\right)} \qquad (n=0)$$

$$A_{ni} = \frac{\int_0^R \int_0^{2\pi} u_0(\rho_0, \vartheta_0)\, J_n\left(\sqrt{\lambda_{ni}}\, \rho_0\right) \rho_0 \cos(n\vartheta_0)\, d\rho_0 \, d\vartheta_0}{\frac{\pi}{2} R^2 \, J_{n+1}^2\left(\sqrt{\lambda_{ni}}\, R\right)} \qquad (n>0)$$

$$B_{ni} = \frac{\int_0^R \int_0^{2\pi} u_0(\rho_0, \vartheta_0)\, J_n\left(\sqrt{\lambda_{ni}}\, \rho_0\right) \rho_0 \sin(n\vartheta_0)\, d\rho_0 \, d\vartheta_0}{\frac{\pi}{2} R^2 \, J_{n+1}^2\left(\sqrt{\lambda_{ni}}\, R\right)} \qquad (n>0)$$

NOTES:

- If $u_0(\rho, \vartheta) = u_0(\rho)$ does not depend on ϑ, then $A_{ni} = B_{ni} = 0$ for $n > 0$

- In the NO-FLUX B.C. case:

$$A_{00} = \frac{\int_0^R \int_0^{2\pi} u_0(\rho_0, \vartheta_0)\, \rho_0 \, d\rho_0 \, d\vartheta_0}{\pi R^2} = \langle u_0(\rho_0, \vartheta_0) \rangle$$

AVERAGE

GREEN's formulation: by substitution of the integrals in the constants A_{mi} & B_{mi} in the solution:

$$u(\rho,\vartheta,t) = \sum_m \sum_i J_m(\sqrt{\lambda_{mi}}\,\rho)\left(A_{mi}\cos m\vartheta + B_{mi}\sin m\vartheta\right) . e^{-\lambda_{mi} D t}$$

$$= \int_0^R \int_0^{2\pi} u_0(\rho_0,\vartheta_0)\, G(\rho,\vartheta,t;\rho_0,\vartheta_0,0)\, \underbrace{\rho_0\, d\rho_0\, d\vartheta_0}_{dA_0 \text{ : AREA}}$$

with GREEN's function:

$$G(\rho,\vartheta,t;\rho_0,\vartheta_0,t_0) = \sum_m \sum_i J_m(\sqrt{\lambda_{mi}}\,\rho_0) J_m(\sqrt{\lambda_{mi}}\,\rho) \cdot \frac{C_{mi}}{R^2}$$

$$\underbrace{\left(\cos m\vartheta_0 \cos m\vartheta + \sin m\vartheta_0 \sin m\vartheta\right)}_{\cos m(\vartheta-\vartheta_0)} \cdot e^{-\lambda_{mi} D (t-t_0)}$$

NOTES:

- G is SHIFT-INVARIANT besides TIME-INVARIANT:

$$G(\rho,\vartheta,t;\rho_0,\vartheta_0,t_0) = G(\rho,\vartheta-\vartheta_0,t-t_0;\rho_0,0,0)$$

- $$C_{mi} = \begin{cases} \dfrac{1}{\pi\; J_1^2(\sqrt{\lambda_{0i}}\,R)}, & m=0 \\[2ex] \dfrac{2}{\pi\; J_{m+1}^2(\sqrt{\lambda_{mi}}\,R)}, & m>0 \end{cases}$$ are CONSTANTS

Examples :

- $\mu_0(\rho_0, \vartheta_0) = 1$ (constant) :

$$\Rightarrow \mu(\rho, \vartheta, t) = \sum_i \frac{2\pi C_{0i}}{R^2} \left(\int_0^R J_0(\sqrt{\lambda_{0i}}\,\rho_0)\,\rho_0\,d\rho_0 \right) J_0(\sqrt{\lambda_{0i}}\,\rho)\, e^{-\lambda_{0i} D t}$$

- $\mu_0(\rho_0, \vartheta_0) = 1 - \frac{\rho_0^2}{R^2}$ (steady state of constant uniform source with zero B.C.) :

$$\Rightarrow \mu(\rho, \vartheta, t) = \sum_i \frac{2\pi C_{0i}}{R^2} \left(\int_0^R J_0(\sqrt{\lambda_{0i}}\,\rho_0) \left(1 - \frac{\rho_0^2}{R^2}\right) \rho_0\,d\rho_0 \right) J_0(\sqrt{\lambda_{0i}}\,\rho) \cdot e^{-\lambda_{0i} D t}$$

where :

$$\int_0^R \rho_0\, J_0(\sqrt{\lambda_{0i}}\,\rho_0)\, d\rho_0 = \frac{R}{\sqrt{\lambda_{0i}}} \cdot J_1(\sqrt{\lambda_{0i}}\, R)$$

and :

$$\int_0^R \rho_0^3\, J_0(\sqrt{\lambda_{0i}}\,\rho_0)\, d\rho_0 =$$

$$\frac{R^2}{\lambda_{0i}} \left(2\, J_2(\sqrt{\lambda_{0i}}\, R) - \sqrt{\lambda_{0i}}\, R \,.\, J_3(\sqrt{\lambda_{0i}}\, R) \right)$$

The same GREEN's formulation extends to non-homogeneous problems as before, e.g:

$$\frac{\partial u}{\partial t} = D\,\Delta u + Q(\rho,\vartheta,t) \quad \text{with} \quad \begin{cases} u(R,\vartheta,t) = u_R(\vartheta,t) & : \text{B.C.} \\ u(\rho,\vartheta,0) = u_0(\rho,\vartheta) & : \text{I.C.} \end{cases}$$

($Q(\rho,\vartheta,t)$ ↑ DRIVING SOURCE)

⇓

$$u(\rho,\vartheta,t) = \int_0^R\int_0^{2\pi}\int_0^t Q(\rho_0,\vartheta_0,t_0)\cdot G(\rho,\vartheta,t;\rho_0,\vartheta_0,t_0)\cdot \underbrace{\rho_0\,d\rho_0\,d\vartheta_0}_{dA_0:\ \text{AREA}}\,dt_0$$

(Q ↓ DRIVING SOURCE; G ↓ Green's)

$$+\int_0^R\int_0^{2\pi} u_0(\rho_0,\vartheta_0)\cdot G(\rho,\vartheta,t;\rho_0,\vartheta_0,0)\cdot \underbrace{\rho_0\,d\rho_0\,d\vartheta_0}_{dA_0:\ \text{AREA}}$$

(u_0 ↓ I.C.; G ↓ Green's @ $t_0 = 0$)

$$-\int_0^{2\pi}\int_0^t u_R(\vartheta_0,t_0)\cdot D\frac{\partial}{\partial\rho_0}G(\rho,\vartheta,t;R,\vartheta_0,t_0)\cdot \underbrace{R\,d\vartheta_0}_{dl_0:\ \text{ARC LENGTH}}\cdot dt_0$$

(u_R ↓ B.C. @ $\rho_0 = R$; $D\frac{\partial}{\partial\rho_0}G$ ↓ Green's OUTFLUX @ $\rho_0 = R$)

FROM POLAR TO CYLINDRICAL COORDINATES :

Homogeneous diffusion in 3-D :

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad ; \quad \text{homogeneous B.C.}$$

Cartesian separation of variables :

Let $u(x, y, z, t) = v(x, y, t) \cdot w(z, t)$

keep time! keep time!

$$\Rightarrow \frac{\partial v}{\partial t} \cdot w + v \cdot \frac{\partial w}{\partial t} = D\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \cdot w + D\, v \cdot \frac{\partial^2 w}{\partial z^2}$$

$$\Rightarrow \underbrace{\frac{\frac{\partial v}{\partial t} - D\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}{v}}_{= \lambda(t)} + \underbrace{\frac{\frac{\partial w}{\partial t} - D\,\frac{\partial^2 w}{\partial z^2}}{w}}_{= -\lambda(t)} = 0$$

function of time only and its complement

CHOOSE $\lambda(t) \equiv 0$ (smart choice : simplifies solution, and avoids instabilities)

$$\Rightarrow \begin{cases} \dfrac{\partial v}{\partial t} = D\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) = D\left(\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial v}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 v}{\partial \theta^2}\right) \\ \dfrac{\partial w}{\partial t} = D\,\dfrac{\partial^2 w}{\partial z^2} \end{cases}$$

→ 2-D CARTESIAN homogeneous solution

→ 2-D POLAR homogeneous solution

→ 1-D LINEAR homogeneous solution