BENG 221: Mathematical Methods in Bioengineering

Lecture 17

Electrostatics

References

http://en.wikipedia.org/wiki/Electrostatics
http://en.wikipedia.org/wiki/Conservative_force
http://en.wikipedia.org/wiki/Coulomb%27s_law
http://en.wikipedia.org/wiki/Gauss_law
http://en.wikipedia.org/wiki/Dielectric
http://en.wikipedia.org/wiki/Electric_displacement_field
http://en.wikipedia.org/wiki/Capacitance
http://en.wikipedia.org/wiki/Dipole

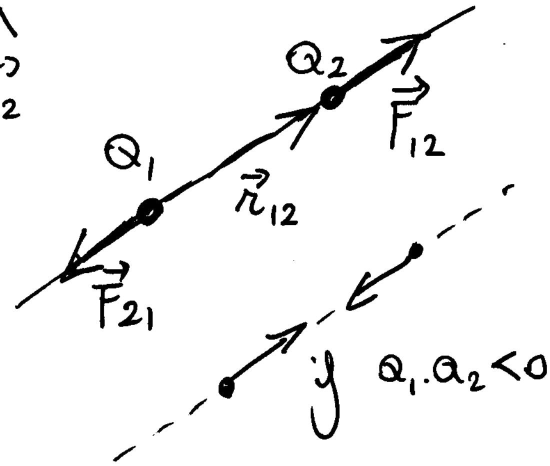
http://farside.ph.utexas.edu/teaching/em/lectures/lectures.html

http://www.numericana.com/answer/maxwell.htm

Paul L. Nunez and Ramesh Srinivasan, Electric Fields of the Brain: The Neurophysics of EEG, Oxford Univ. Press, 2006.

Coulomb's law

$$\vec{F}_{12} = -\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 . Q_2}{|\vec{r}_{12}|^2} . \hat{\vec{r}}_{12}$$

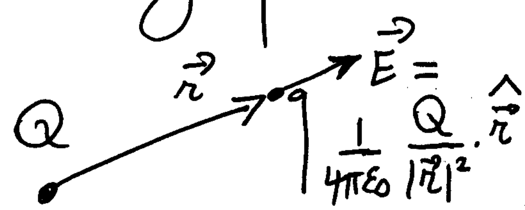


with $\hat{\vec{r}}_{12}$ = unit vector in direction $\vec{r}_{12}$

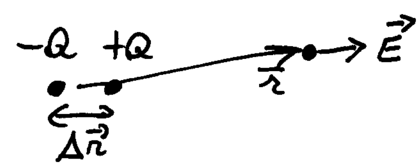
Electric field:

$\vec{F}(\vec{r}) = q . \vec{E}(\vec{r})$ force acting on charge q at $\vec{r}$

- Q at origin (monopole)



- $+Q, -Q$ centered near origin (dipole) @ distance $\Delta\vec{r}$



- multiple Q_i

$$\vec{E}(\vec{r}_q) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{Q_i}{|\vec{r}_{iq}|^2} . \hat{\vec{r}}_{iq} \quad , \quad \vec{r}_{iq} = \vec{r}_q - \vec{r}_i$$

- distributed charge density:

$$\vec{E}(\vec{r}_q) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho(\vec{r}_i)}{|\vec{r}_{iq}|^2} . \hat{\vec{r}}_{iq} . dV(\vec{r}_i)$$

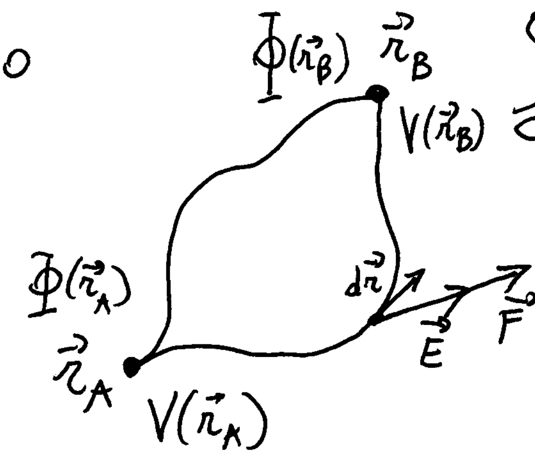
$\vec{F}(\vec{r}) = q \cdot \vec{E}(\vec{r})$ is a CONSERVATIVE FIELD

$$\vec{F} = -\vec{\nabla} \cdot \Phi \quad \longleftrightarrow \quad \vec{E} = -\vec{\nabla} \cdot V$$

FORCE — POTENTIAL ENERGY — ELECTRIC FIELD — ELECTRIC POTENTIAL

$$\vec{\nabla} \times \vec{F} = 0 \qquad \vec{\nabla} \times \vec{E} = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \qquad \oint_C \vec{E} \cdot d\vec{r} = 0$$



$$\int_{\vec{r}_A}^{\vec{r}_B} \vec{F} \cdot d\vec{r} = \Phi(\vec{r}_A) - \Phi(\vec{r}_B)$$

$$\int_{\vec{r}_A}^{\vec{r}_B} \vec{E} \cdot d\vec{r} = V(\vec{r}_A) - V(\vec{r}_B)$$

ELECTRIC POTENTIAL DIFFERENCE

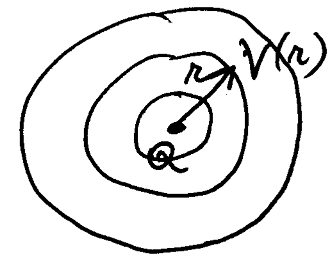
↓ WORK A → B
independent of path from A → B

↓ POTENTIAL ENERGY @ A

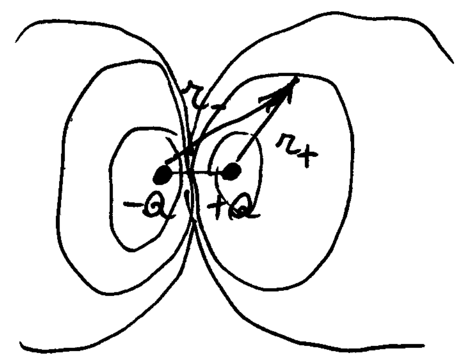
↓ POTENTIAL ENERGY @ B

Electric Potential:

- monopole $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\vec{r}} \longleftrightarrow V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$



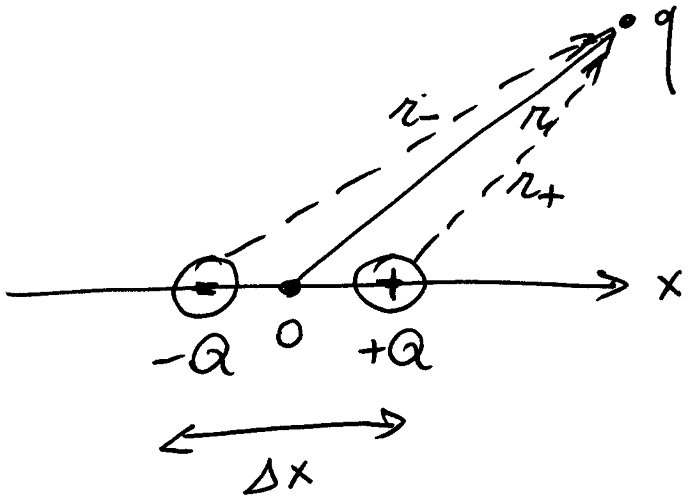
- dipole $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{\hat{\vec{r}}_+}{r_+^2} - \frac{\hat{\vec{r}}_-}{r_-^2}\right) \longleftrightarrow V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$

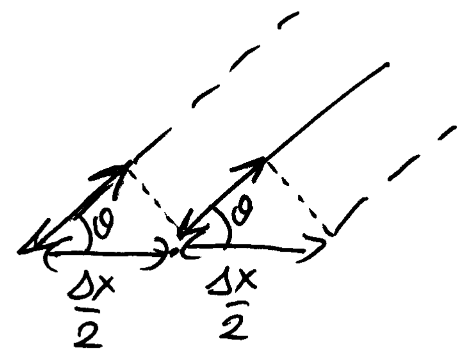


- general case $\vec{E}(\vec{r}_q) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{|\vec{r}_{iq}|^2}\cdot\hat{\vec{r}}_{iq} \leftrightarrow V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{|\vec{r}_{iq}|}$

$$\vec{E}(\vec{r}_q) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(\vec{r}_i)}{|\vec{r}_{iq}|^2}\cdot\hat{\vec{r}}_{iq}\, dV_{\vec{r}_i} \leftrightarrow V(\vec{r}_q) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\vec{r}_i)}{|\vec{r}_{iq}|}\, dV_{\vec{r}_i}$$

Example: electrical dipole



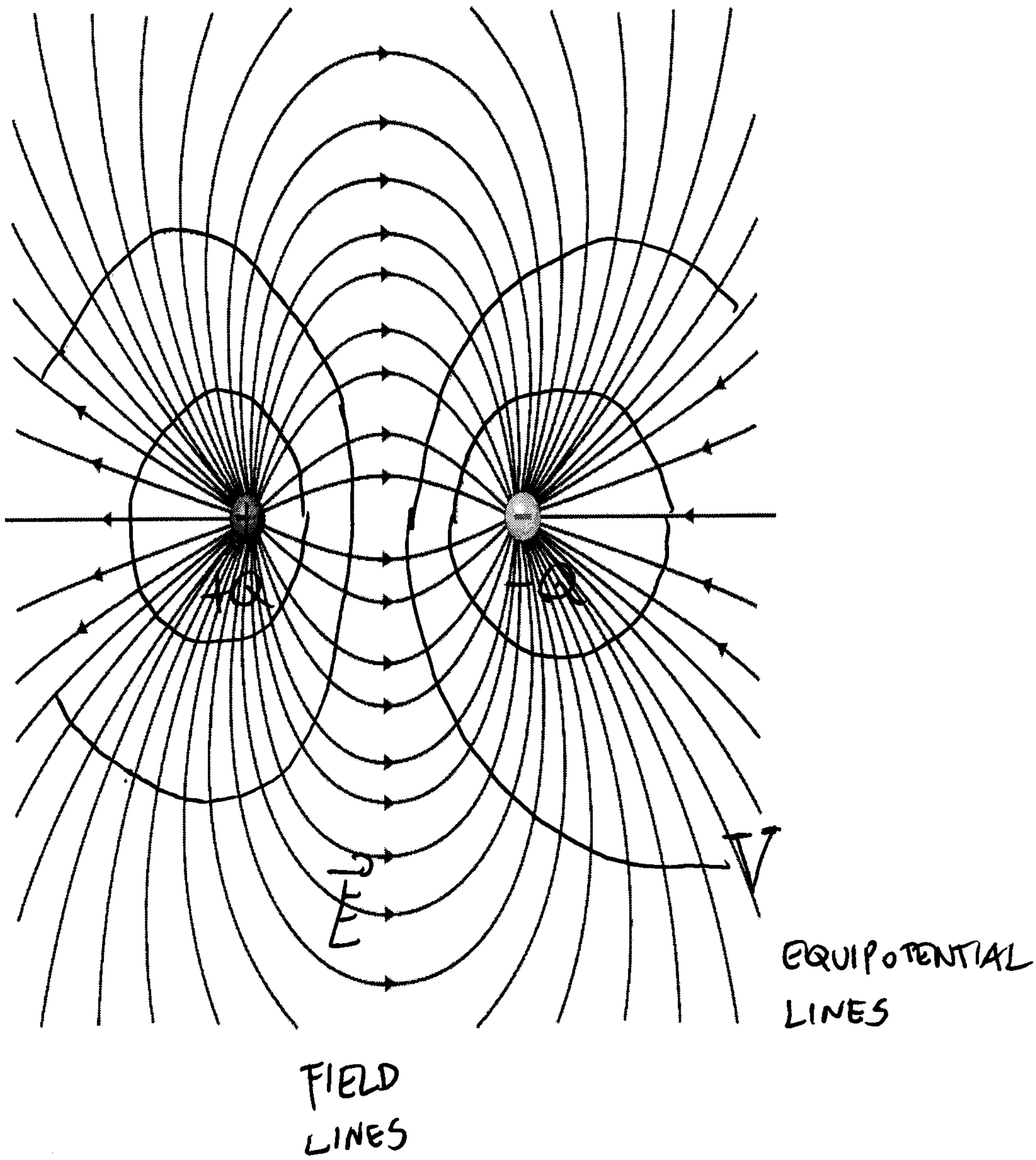


$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$ where $$r_+ \approx r - \frac{\Delta x}{2}\cos\vartheta$$ $$r_- \approx r + \frac{\Delta x}{2}\cos\vartheta$$

$$\Rightarrow V \approx \frac{Q}{4\pi\varepsilon_0}\Delta x \cdot \frac{\cos\vartheta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{\vec{r}}}{r^2}$$

with $\vec{p} = Q\,\Delta x \cdot \hat{\vec{x}}$

electric dipole moment

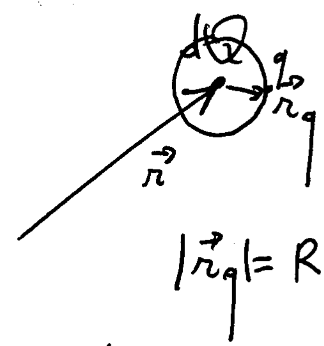

+Q
-Q
$\vec{E}$
V
EQUIPOTENTIAL LINES
FIELD LINES

Gauss's law

$$\operatorname{div}\vec{E} = \vec{\nabla}\cdot\vec{E} = \lim_{V\to 0} \frac{1}{V} \oiint_{S(V)} \vec{E}\cdot\vec{n}S$$

Let $S(V)$ = a sphere around monopole

$$dQ = \rho(\vec{r})\,.\,dV$$



where $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r_q^2}\,\hat{\vec{r}}_q$$

$$\Rightarrow \operatorname{div}\vec{E} = \lim_{\substack{R\to 0\\ dV\to 0}} \frac{1}{dV}\cancel{4\pi R^2}\,\frac{1}{\cancel{4\pi}\varepsilon_0}\frac{dQ}{\cancel{R^2}} = \frac{1}{\varepsilon_0}\frac{dQ}{dV} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$$

Integral form: $$\oiint_{S(V)} \vec{E}\cdot\vec{n}S = \iiint_V \operatorname{div}\vec{E}\,dV = \frac{1}{\varepsilon_0}\iiint_V \rho\,dV$$

Poisson : $\vec{E} = -\vec{\nabla} \cdot V$ and

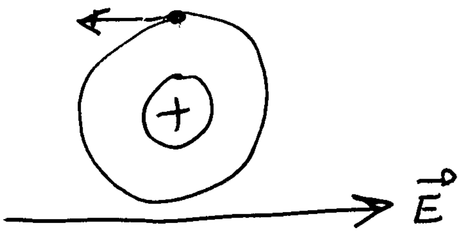
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \Delta V = |\vec{\nabla}|^2 V = -\frac{\rho}{\varepsilon_0}$$

Laplace : $\rho \equiv 0$

$$\Rightarrow \quad \Delta V = 0$$

ELECTRIC FIELD IN DIELECTRICS (INSULATED MATERIALS)



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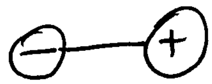


polarization

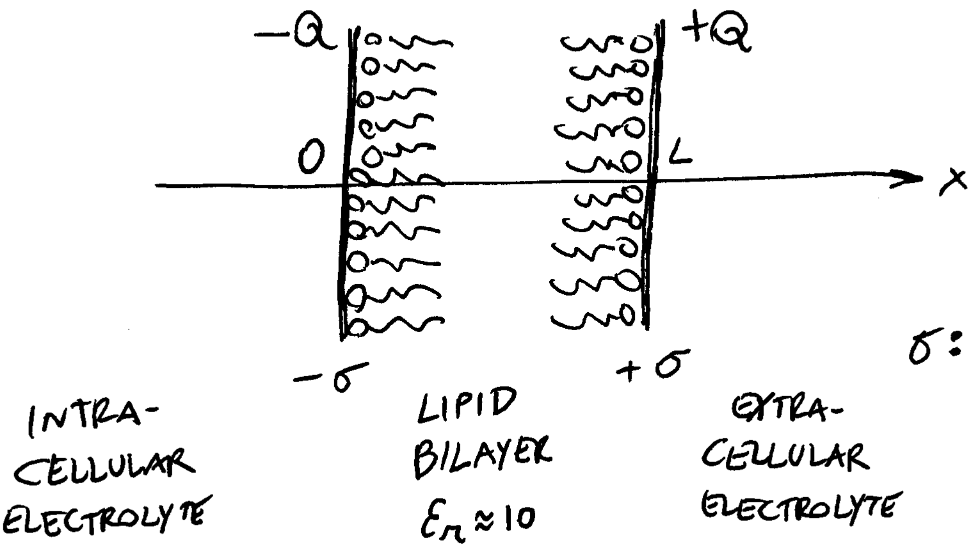
$$\varepsilon_0 \vec{\nabla}.\vec{E} = \rho_{total} = \rho_{free} + \rho_{pol.}$$

where $\rho_{pol.} = -\vec{\nabla}.\vec{P}$ and $\vec{P} = \varepsilon_0 \chi \vec{E}$

$$\Rightarrow \underbrace{\varepsilon_0 . (1 + \chi)}_{\varepsilon} . \vec{\nabla} . \vec{E} = \rho_{free}$$

$1 + \chi = \varepsilon_r$ relative permittivity

Example: CAPACITOR, such as CELL MEMBRANE



σ: charge surface density $\left[\frac{C}{m^2}\right]$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$$

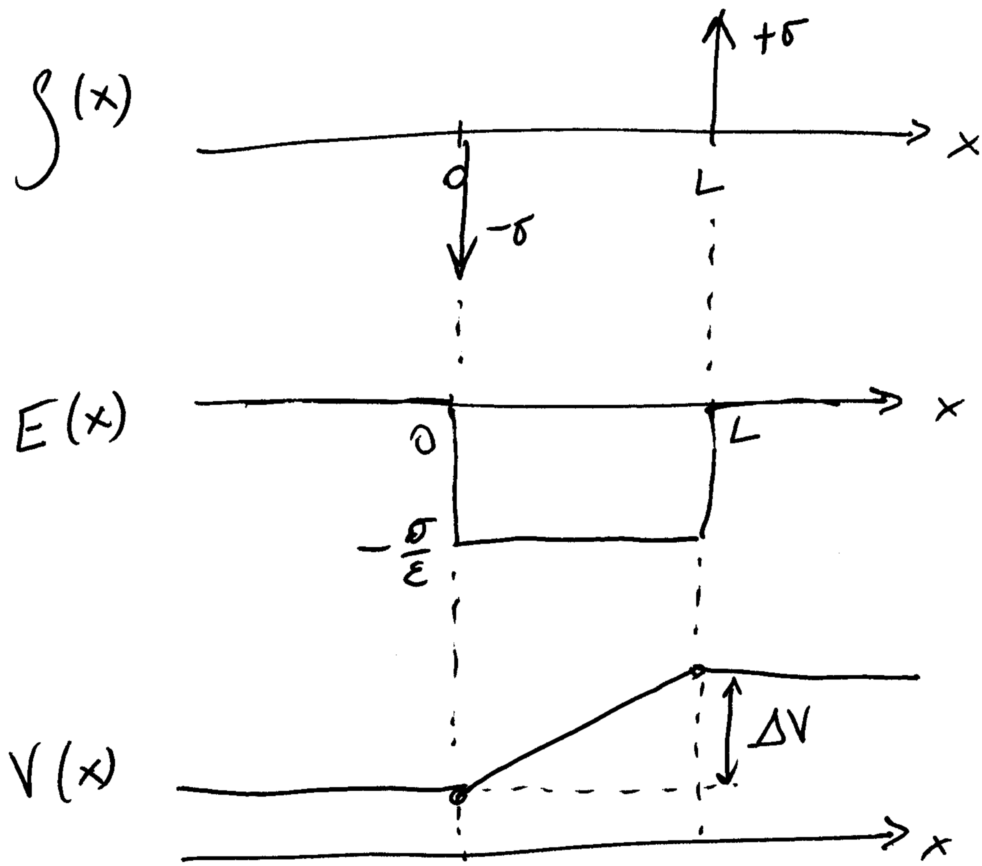
$$\Downarrow$$

$$\frac{d}{dx}E(x) = \frac{\rho(x)}{\varepsilon}$$

$$E(x) = \frac{1}{\varepsilon}\int_{-\infty}^{x} \rho \, dx \quad \text{with} \quad \rho(x) = -\sigma\,\delta(x) + \sigma\,\delta(x-L)$$

where $\displaystyle\int_{x_0-\xi}^{x_0+\xi} \delta(x-x_0)\,dx = 1$ for all ξ

$$\Rightarrow \quad E(x) = \begin{cases} E(-\infty) = 0 & \ldots \text{ for } x<0 \ldots\ldots \\ -\dfrac{\sigma}{\varepsilon} & \ldots\ldots\ 0<x<L\ ; \\ 0 & \ldots\ldots\ x>L \ldots\ldots \end{cases} \qquad V(x) = \begin{cases} V_{-\infty} = V_0 \\ V_0 + \dfrac{\sigma}{\varepsilon}x \\ V_0 + \dfrac{\sigma}{\varepsilon}L \end{cases}$$



$$\Delta V = V(L) - V(0) = \frac{\sigma}{\varepsilon} \cdot L$$

A = area of plates $\Rightarrow$ $Q = \sigma \cdot A = \Delta V \cdot \underbrace{\frac{\varepsilon}{L} \cdot A}_{C}$

$$\Rightarrow C = \varepsilon \frac{A}{L} = \varepsilon_0 \varepsilon_r \frac{A}{L}$$

e.g.: LIPID BILAYER

cell membrane capacitance:

$\varepsilon_0 = 8.8 \; 10^{-12} \; F/m$

$\varepsilon_r \approx 10$

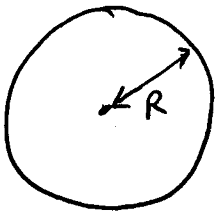
$d \approx 10 \; nm = 10^{-8} \; m$

$$\Rightarrow \quad \frac{C}{A} = \frac{\varepsilon_0 \varepsilon_r}{L} = \frac{8.8 \; 10^{-12} \cdot 10}{10^{-8}} \frac{F}{m^2}$$

$$= 8.8 \; 10^{-3} \frac{F}{m^2} \approx 1 \mu F/cm^2$$

Example : Conducting ball of radius R

(spherical model of a cell)



$V(\vec{r}) = V(r)$ because symmetry

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon} \quad \text{with} \quad \rho = \sigma \, \delta(r-R)$$ because of conductor

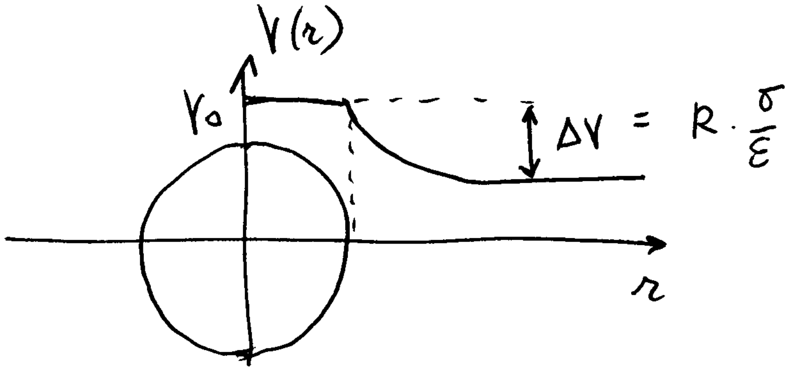
$\Downarrow$ where $Q = \underbrace{4\pi R^2}_{A} \cdot \sigma$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \cdot E \right) = \frac{\rho}{\varepsilon}$$

$$\frac{d}{dr} \left(r^2 . E \right) = r^2 \frac{\rho}{\varepsilon}$$

$$E = \frac{1}{r^2} \int_0^r r'^2 \frac{\rho}{\varepsilon} \, dr' = \begin{cases} 0 & 0 < r < R \\ \frac{\sigma}{\varepsilon} & r = R \\ \frac{R^2}{r^2} \cdot \frac{\sigma}{\varepsilon} & r > R \end{cases}$$

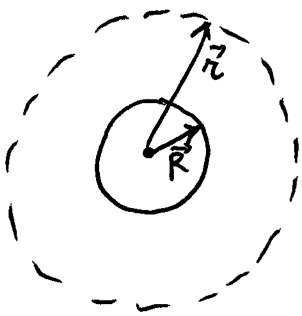
$$\Rightarrow V(r) = V_0 - \int_0^r E(r')\,dr' = \begin{cases} V_0, & 0 < r < R \\ V_0 + \left(\frac{R^2}{r} - R\right)\frac{\sigma}{\varepsilon}, & r > R \end{cases}$$



$$\Rightarrow Q = 4\pi R^2 \cdot \sigma = 4\pi R^{\cancel{2}} \frac{\Delta V \varepsilon}{\cancel{R}} = \underbrace{4\pi R \cdot \varepsilon}_{C} \cdot \Delta V$$

$$\Rightarrow C = \varepsilon\, 4\pi R = \varepsilon_0 \cdot \varepsilon_r\, 4\pi R$$

Alternative solution: Gauss in integral form



$$\oiint_{S(r)} \vec{E} \cdot \vec{n}\, dA = \frac{1}{\varepsilon} \iiint_{V(r)} \rho\, dV$$

$$\Downarrow$$

$$E(r) \cdot 4\pi r^2 = \begin{cases} 0 & \text{if } r < R \\ \frac{1}{\varepsilon} Q & \text{if } r \geq R \end{cases}$$

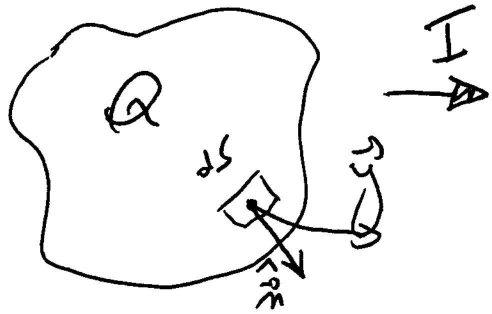
$$E(r) = \begin{cases} 0 & \text{if } r < R \\ \frac{1}{4\pi\varepsilon} \frac{1}{r^2} \cdot Q & \text{if } r > R \end{cases}$$

$$V(r) - V(R) = -\int_0^r E(r')\, dr' = \begin{cases} 0 & \text{if } r < R \\ \frac{1}{4\pi\varepsilon}\left(\frac{1}{r} - \frac{1}{R}\right) Q & \text{otherwise} \end{cases}$$

$$\underbrace{V(\infty) - V(R)}_{-\Delta V} = -\frac{1}{4\pi R} \frac{1}{R} \cdot Q$$

$$\text{or } C = \varepsilon 4\pi R = \varepsilon_0 \varepsilon_r 4\pi R$$

Conservation of charge:



$$\frac{dQ}{dt} = -I$$

$\frac{dQ}{dt}$: CHARGE ACCUMULATION INTERNALLY

I: OUTWARD CURRENT

- Integral notation:

$$\frac{\partial}{\partial t}\iiint_{V} \rho \, dV = -\oiint_{S(V)} \vec{j}\,.\,\vec{n}\,.\,S$$

- Differential form:

$$\frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot\vec{j}$$

ELECTRIC FIELD IN & NEAR CONDUCTORS

Inside conductor: $\vec{j} = \sigma \vec{E}$ Ohm's law

$\text{div}\, \vec{j} + \frac{\partial \rho}{\partial t} = 0$ Lorenz gauge (transport equation)

$$\Rightarrow \sigma \,\text{div}\, \vec{E} + \frac{\partial \rho}{\partial t} = 0$$

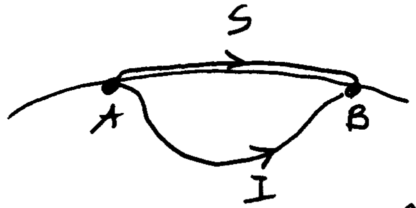
$$\underset{\text{Gauss}}{\Rightarrow} \sigma \frac{\rho}{\varepsilon_0} + \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \rho(\vec{r},t) = \rho(\vec{r},0) \cdot e^{-\frac{\sigma}{\varepsilon_0} t}$$

$\Rightarrow \rho(\vec{r},t) \approx 0$ INSIDE CONDUCTOR @ EQUILIBRIUM

$\Rightarrow$ $\underline{\vec{E}(\vec{r},t) \approx 0 \text{ inside, and all charge = on the boundary}}$

Boundary :

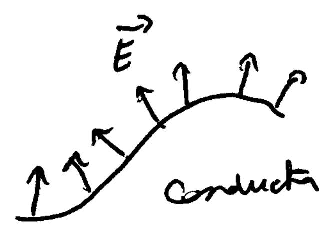


S : surface
I : inside

$$\int_S \vec{E} \cdot d\vec{l} = \int_I \vec{E} \cdot d\vec{l} = 0$$

$\Rightarrow \vec{E} \cdot d\vec{l} = 0$ for all $d\vec{l}$ on the surface, or

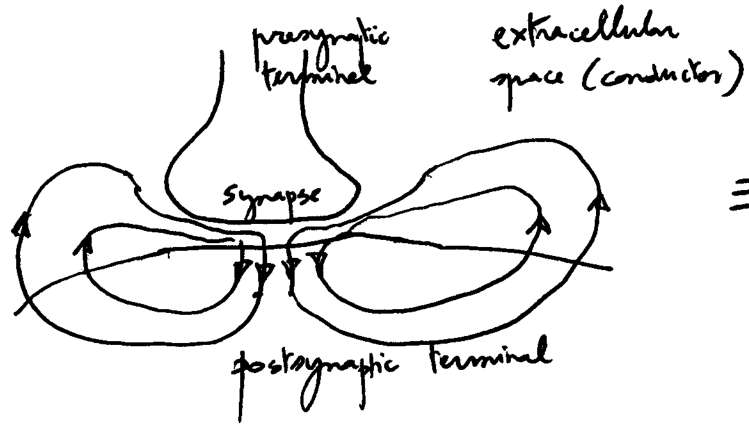
$$\vec{E} \perp S$$



$\underline{\vec{E} \text{ is perpendicular to the surface}}$

CONDUCTOR WITH INTERNAL CURRENTS

e.g. : EEG, ECG, EMG, EOG, ...



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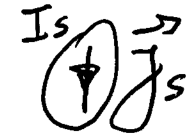


monopole current source + diffuse return current

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \approx 0$$

Transport in conductor

$$\vec{J} = \sigma\vec{E} + \vec{J}^s$$

Ohm's law, plus monopole sources

$$\Downarrow$$

$$\vec{\nabla}\cdot(\sigma\vec{\nabla}V) = +\vec{\nabla}\cdot\vec{J}^s$$

or $$\Delta V = \frac{1}{\sigma}\cdot \operatorname{div}\vec{J}^s = \frac{1}{\sigma}\cdot I_s\cdot\delta(\vec{r}-\vec{r}_0)$$

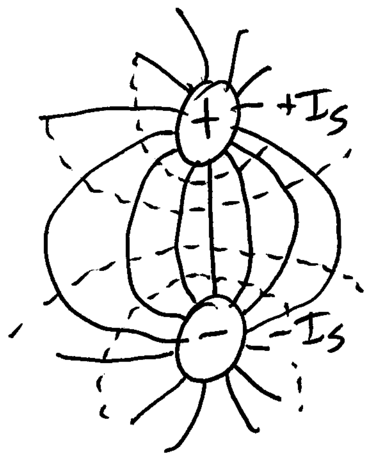
monopole source @ $\vec{r}_0$

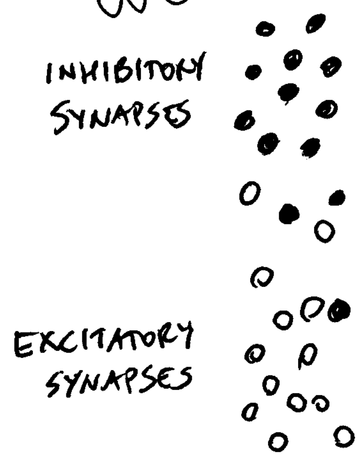
$$\Rightarrow\ V(\vec{r},t) = \frac{1}{4\pi\sigma}\cdot\frac{I_s(t)}{|\vec{r}-\vec{r}_0|}$$

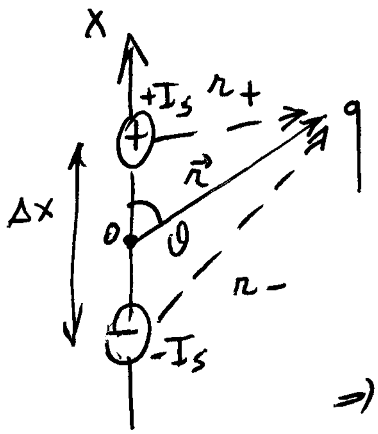
and superposition for multiple monopoles

Dipole revisited — now due to current monopoles in conductor

Current dipole: aggregate of population of synapses







$$V = \frac{I_S}{4\pi\sigma} \cdot \left(\frac{1}{r_+} - \frac{1}{r_-} \right)$$

$$\Rightarrow V \cong \frac{I_S}{4\pi\sigma} \cdot \frac{\Delta x \cos\vartheta}{r^2} = \frac{1}{4\pi\sigma} \cdot \frac{\vec{p} \cdot \hat{\vec{r}}}{r^2}$$

$$\vec{p} = I_S \cdot \Delta x \cdot \hat{\vec{x}}$$

current dipole moment

e.g.: $\sigma \approx 0.2 \frac{1}{\Omega m}$ in cortex

$I_S \approx 1\, nA$ typical microcolumn

$\Delta x \approx 1\, mm$

$$\Rightarrow V \approx \frac{I_S}{4\pi\sigma} \frac{\Delta x}{r^2} \cong \frac{10^{-9}}{12.5 \times 0.2} \cdot \frac{10^{-3}}{(10^{-1})^2} V \cong 1.4\ 10^{-10} V = 0.14\, nV$$

$\downarrow$

$r \approx 10$ cm to scalp

Small! → requires low-noise instrumentation