BENG 221: Mathematical Methods in Bioengineering

Lecture 18

Electrodynamics in Inert and Live Media

References

Haberman APDE, Ch. 12.

http://en.wikipedia.org/wiki/Electromagnetic_radiation

http://en.wikipedia.org/wiki/Method_of_characteristics

http://en.wikipedia.org/wiki/D%27Alembert%27s_formula

http://en.wikipedia.org/wiki/Near_and_far_field

http://farside.ph.utexas.edu/teaching/em/lectures/lectures.html

http://www.numericana.com/answer/maxwell.htm

Paul L. Nunez and Ramesh Srinivasan, Electric Fields of the Brain: The Neurophysics of EEG, Oxford Univ. Press, 2006.

In Summary:

Gauss: $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ or: $\displaystyle\oiint_{S(V)} \vec{E}\cdot\vec{n}\, dS = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$

Faraday: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ or: $\displaystyle\oint_{C(S)} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot\vec{n}\, dS$

Gauss for magnetism: $\vec{\nabla}\cdot\vec{B} = 0$ or: $\displaystyle\oiint_{S(V)} \vec{B}\cdot\vec{n}\, dS = 0$

Ampère with Maxwell correction: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$ or: $\displaystyle\oint_{C(S)} \vec{B}\cdot d\vec{l} = \mu_0\iint_S \vec{J}\cdot\vec{n}\, dS + \mu_0\varepsilon_0\frac{\partial}{\partial t}\iint_S \vec{E}\cdot\vec{n}\, dS$

In dielectric media: $\underbrace{\varepsilon_0}_{\text{VACUUM PERMITTIVITY}} \rightarrow \varepsilon = \varepsilon_0\cdot\underbrace{\varepsilon_r}_{\text{RELATIVE PERMITTIVITY}} = \varepsilon_0(1+\chi)$

In magnetic susceptible media: $\underbrace{\mu_0}_{\text{VACUUM SUSCEPTIBILITY}} \rightarrow \mu = \mu_0\cdot\underbrace{\mu_r}_{\text{RELATIVE SUSCEPTIBILITY}} = \mu_0(1+\chi_m)$

Wave Propagation:

In vacuum: $\rho = 0$ and $\vec{j} = 0$ no charges/currents

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

since $\vec{\nabla} \times \left(\vec{\nabla} \times \begin{matrix}\vec{E} \\ \vec{B}\end{matrix}\right) = \vec{\nabla}\left(\underbrace{\vec{\nabla} \cdot \begin{matrix}\vec{E} \\ \vec{B}\end{matrix}}_{=0}\right) - \vec{\nabla}^2 \begin{pmatrix}\vec{E} \\ \vec{B}\end{pmatrix}$

$$\| $$

$$\vec{\nabla} \times \frac{\partial}{\partial t}\begin{pmatrix} -\vec{B} \\ \mu_0 \varepsilon_0 \vec{E}\end{pmatrix}$$

$$\|$$

$$-\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\begin{pmatrix}\vec{E} \\ \vec{B}\end{pmatrix}$$

$$\Rightarrow \quad \vec{\nabla}^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

(and $\mu_0 \varepsilon_0 \rightarrow \mu \varepsilon$ in other media)

$\mu_0 \varepsilon_0 = \frac{1}{c^2}$ or $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

POTENTIALS REVISITED : EM WAVES

STATIC: $\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} V$

$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$

where $\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad V = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dV$

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J} \quad \Rightarrow \quad \vec{A} = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} dV$$

$\vec{\nabla} \cdot \vec{A} = 0$

Coulomb gauge and $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$

DYNAMIC: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

and $\vec{B} = \vec{\nabla} \times \vec{A}$ remains

now: $\vec{\nabla}^2 V = \vec{\nabla} \cdot \vec{\nabla} V = \vec{\nabla} \cdot \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right)$

$$= -\frac{\rho}{\varepsilon_0} + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right)$$

Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ does not work here...

Lorenz gauge : $\vec{\nabla}\cdot\vec{A} = -\varepsilon_0\mu_0 \dfrac{\partial V}{\partial t}$

$$\Rightarrow \quad \vec{\nabla}^2 V - \varepsilon_0\mu_0 \frac{\partial^2}{\partial t^2} V = -\frac{\rho}{\varepsilon_0}$$ WAVE EQUATION for V

$$\underbrace{\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\Box} V = -\frac{\rho}{\varepsilon_0} \Rightarrow V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\, dV$$

$c = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}}$ speed of light

Also, $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$

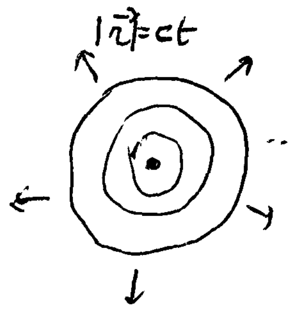
$$\vec{\nabla}\times\vec{B} = \cancel{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}} + \mu_0\vec{j} \qquad -\varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{\nabla}V = \cancel{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}} + \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Rightarrow \quad \vec{\nabla}^2\vec{A} - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{j}$$ WAVE EQUATION for $\vec{A}$

$$\underbrace{\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)}_{\Box}\vec{A} = -\mu_0\vec{j} \Rightarrow \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{j}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\, dV$$

$t - \dfrac{|\vec{r}-\vec{r}\,'|}{c}$: RETARDED POTENTIALS

radial propagation @ c



RETARDED FIELDS:

$$\begin{cases} \vec{E} = -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases} \quad \text{where} \quad \begin{cases} V = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\iiint \dfrac{[\rho]}{R}\, dV \\ \vec{A} = \dfrac{\mu_0}{4\pi} \displaystyle\iiint \dfrac{[\vec{j}]}{R}\, dV \end{cases}$$

and
$$[\rho] = \rho\left(\vec{r}\,', t - \frac{R}{c}\right)$$
$$[\vec{j}] = \vec{j}\left(\vec{r}\,', t - \frac{R}{c}\right)$$
RETARDED POTENTIALS
$$\vec{R} = \vec{r} - \vec{r}\,'$$
$$R = |\vec{R}|$$

Since
$$\vec{\nabla} \frac{[\rho]}{R} = -\frac{[\rho]}{R^2} \cdot \hat{\vec{R}} - \frac{1}{c}\left[\frac{\partial \rho}{\partial t}\right] \frac{1}{R}\, \hat{\vec{R}}$$
$$\vec{\nabla} \times \frac{[\vec{j}]}{R} = \frac{[\vec{j}] \times \hat{\vec{R}}}{R^2} + \frac{1}{c}\left[\frac{\partial \vec{j}}{\partial t}\right] \times \frac{\hat{\vec{R}}}{R}$$
$$\left(= -\frac{\hat{\vec{R}} \times [\vec{j}]}{R^2} - \frac{1}{c}\, \frac{\hat{\vec{R}} \times \left[\frac{\partial \vec{j}}{\partial t}\right]}{R} \right)$$

$$\Rightarrow \quad \vec{E} = \frac{1}{4\pi\varepsilon_0} \iiint \left(\underbrace{\frac{[\rho]}{R^2}\, \hat{\vec{R}}}_{\text{COULOMB}} + \frac{1}{c}\, \frac{\left[\frac{\partial \rho}{\partial t}\right]}{R} \cdot \hat{\vec{R}} - \frac{1}{c^2}\, \frac{\left[\frac{\partial \vec{j}}{\partial t}\right]}{R} \right) dV$$

$$\vec{B} = \frac{\mu_0}{4\pi} \iiint \left(\underbrace{\frac{[\vec{j}] \times \hat{\vec{R}}}{R^2}}_{\text{BIOT-SAVART}} + \frac{1}{c}\, \frac{\left[\frac{\partial \vec{j}}{\partial t}\right] \times \hat{\vec{R}}}{R} \right) dV$$

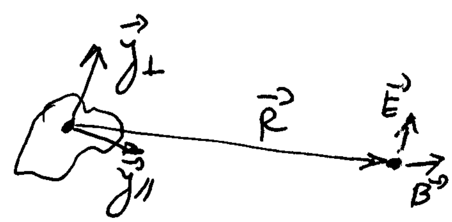
ELECTROSTATIC $\propto \dfrac{1}{R^2}$ ELECTRODYNAMIC $\propto \dfrac{1}{R}$

- Near field: $R \ll \lambda \approx c / \frac{\partial}{\partial t}$

$$\Rightarrow \begin{cases} \vec{E} = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\iiint \dfrac{\rho}{R^2}\, \hat{\vec{R}}\, dV & \text{COULOMB} \\[2ex] \vec{B} = \dfrac{\mu_0}{4\pi} \displaystyle\iiint \dfrac{\vec{j} \times \hat{\vec{R}}}{R^2}\, dV & \text{BIOT-SAVART} \end{cases}$$

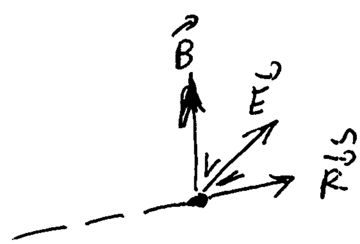
- Far field: $R \gg \lambda \approx c / \frac{\partial}{\partial t}$

$$\begin{cases} \vec{E} \simeq -\dfrac{1}{4\pi\varepsilon_0} \dfrac{1}{c^2 R_*} \cdot \displaystyle\iiint \left[\dfrac{\partial \vec{j}_\perp}{\partial t}\right] dV \\[2ex] \vec{B} \simeq \dfrac{1}{4\pi\varepsilon_0} \dfrac{1}{c^3 R_*} \cdot \displaystyle\iiint \left[\dfrac{\partial \vec{j}_\perp}{\partial t}\right] dV \times \hat{\vec{R}}_* \end{cases}$$



$$\Rightarrow \frac{|\vec{E}|}{|\vec{B}|} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\vec{E} \cdot \vec{B} = 0$$
$$\vec{E} \cdot \hat{\vec{R}} = 0$$
$$\vec{B} \cdot \hat{\vec{R}} = 0$$



and $$\iiint \left[\frac{\partial \vec{j}_\perp}{\partial t}\right] dV = \psi\left(t - \frac{R}{c}\right) \cdot \hat{\vec{j}}_\perp$$

EM WAVES

Maxwell's Equations in Free Space / Lossless Media

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

or $$\vec{\nabla}^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \qquad \mu_0\varepsilon_0 = \frac{1}{c_0^2} \quad \underline{\text{Free Space}}$$

$$\vec{\nabla}^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

or $\mu_0 \to \mu$, $\varepsilon_0 \to \varepsilon$; $\mu\varepsilon = \frac{1}{c^2}$ <u>Lossless Media</u>

or $$\Box\vec{E} = 0 \qquad \text{with} \quad \Box = \vec{\nabla}^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}$$

$$\Box\vec{B} = 0$$

D'Alemenbertian operator

General Solution:

- Planar waves: $$\begin{cases}\vec{E} = \vec{E}_0 \cdot \Psi(\hat{\vec{k}}\cdot\vec{r} - c_0 t) \\ \vec{B} = -\frac{1}{c_0}\vec{E}\times\hat{\vec{k}}\end{cases}$$

- Radial waves: $$\begin{cases}\vec{E} = \vec{E}_0 \cdot \frac{1}{|\vec{r}'|}\Psi(|\vec{r}'| - c_0 t) \\ \vec{B} = -\frac{1}{c_0}\vec{E}\times\hat{\vec{r}}'\end{cases} \qquad \vec{r}' = \vec{r} - \vec{r}_0$$

Maxwell's Equations in Lossy Media:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$\rho = 0$ and $\vec{J} = \sigma\cdot\vec{E}$ (Ohm's Law)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\begin{bmatrix}\vec{E}\\ \vec{B}\end{bmatrix}\right) = \vec{\nabla}\cdot\Big(\underbrace{\vec{\nabla}\cdot\begin{bmatrix}\vec{E}\\ \vec{B}\end{bmatrix}}_{=0}\Big) - \vec{\nabla}^2\cdot\begin{bmatrix}\vec{E}\\ \vec{B}\end{bmatrix}$$

$$= \frac{\partial}{\partial t}\begin{bmatrix}-(\vec{\nabla}\times\vec{B})\\ \mu\varepsilon(\vec{\nabla}\times\vec{E})\end{bmatrix} + \mu\begin{bmatrix}0\\ \vec{\nabla}\times\vec{J}\end{bmatrix}$$

$$= -\frac{\partial^2}{\partial t^2}\begin{bmatrix}\mu\varepsilon\,\vec{E}\\ \mu\varepsilon\,\vec{B}\end{bmatrix} + \begin{bmatrix}-\mu\frac{\partial}{\partial t}\vec{J}\\ \mu\vec{\nabla}\times\vec{J}\end{bmatrix}$$

$$\underset{\vec{J}=\sigma\vec{E}}{\Longrightarrow} \quad \vec{\nabla}^2\vec{E} = \mu_0\sigma\frac{\partial}{\partial t}\vec{E} + \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E}$$

and $\frac{\partial}{\partial t}\vec{B} = -\vec{\nabla}\times\vec{E}$

or $\vec{B} = -\int\vec{\nabla}\times\vec{E}\,dt$

$\lim_{\sigma\to 0}$: WAVE EQUATION

$\lim_{\sigma\to\infty}$: DIFFUSION EQUATION

Radio-Frequency EM Wave Absorption in Biological Tissue

In this problem we study the absorption of radio-frequency (RF) electromagnetic (EM) waves in tissue.

- Consider propagation of harmonic EM waves with radial frequency $\omega = 2\pi f$ through a uniform lossy medium with volume conductivity σ, dielectric constant ϵ, and magnetic permeability μ, of the following form:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_0 \, e^{j\mathbf{k}\cdot\mathbf{r} - j\omega t} \\ \mathbf{B} &= \mathbf{B}_0 \, e^{j\mathbf{k}\cdot\mathbf{r} - j\omega t}\end{aligned}$$

  with wave vector $\mathbf{k}$, electric field strength $\mathbf{E}_0$, and magnetic field strength $\mathbf{B}_0$. Find conditions on $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ for which the waves satisfy Maxwell's equations in the medium. Show that for non-zero volume conductivity σ, the wave vector $\mathbf{k}$ must be complex-valued. Interpret the real and imaginary parts of the wave vector in terms of their effect on wave propagation. In the limit of high frequency, find an expression for the the wave decay space constant ξ defined as the distance over which the field amplitude decays one e-fold.

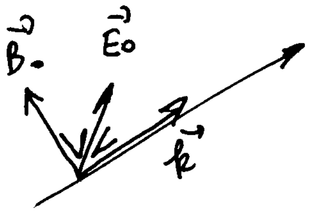
- Show that biological tissue of thickness greater than ξ absorbs at least 86% of incident high-frequency EM radiation. Quantify what is this high-frequency range, and what is this effective thickness of the absorbing tissue layer, for typical values of volume conductivity (such as in the brain), dielectric permittivity (dominated by water), and magnetic permeability (non-paramagnetic) in live tissue.

$$\begin{cases} \vec{E} = \vec{E}_0 \, e^{j\vec{k}\cdot\vec{r} - j\omega t} \\ \vec{B} = \vec{B}_0 \, e^{j\vec{k}\cdot\vec{r} - j\omega t} \end{cases} \quad \text{in} \quad \begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon} = 0 \; ; \; \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \; ; \; \vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t} \end{cases}$$

$$\vec{\nabla} = j\vec{k} \quad \text{and} \quad \frac{\partial}{\partial t} = -j\omega$$

$$\Rightarrow \begin{cases} j\vec{k}\cdot\vec{E}_0 = 0 \; ; \; j\vec{k}\times\vec{E}_0 = j\omega\,\vec{B}_0 \\ j\vec{k}\cdot\vec{B}_0 = 0 \; ; \; j\vec{k}\times\vec{B}_0 = (\mu\sigma - \underbrace{\mu\varepsilon}_{=\frac{1}{c^2}} j\omega)\,\vec{E}_0 \end{cases}$$

$$\Rightarrow \begin{array}{l} \vec{k}\perp\vec{E}_0 \\ \vec{k}\perp\vec{B}_0 \\ \vec{E}_0\perp\vec{B}_0 \end{array}$$



$$\vec{k}\times\vec{k}\times\vec{E}_0 = -\vec{k}\cdot\vec{k}\;\vec{E}_0 = \omega\left(-\frac{\omega}{c^2} - j\mu\sigma\right)\vec{E}_0$$

$$\Rightarrow \vec{k}\cdot\vec{k} = \frac{\omega^2}{c^2} + j\omega\mu\sigma$$

$\Rightarrow \vec{k}$ is complex valued when $\sigma \neq 0$

$$\vec{k} = (k_r + jk_i)\,.\,\hat{\vec{k}}$$

$$\Rightarrow e^{j\vec{k}\cdot\vec{r} - j\omega t} = \underbrace{e^{j\vec{k}_r\cdot\vec{r} - j\omega t}}_{\text{WAVE PROPAGATION}} \cdot \underbrace{e^{-\vec{k}_i\cdot\vec{r}}}_{\text{WAVE DECAY}}$$

wavelength: $\lambda = \frac{2\pi}{k_r}$ space constant: $\xi = \frac{1}{k_i}$

$$k_r + j k_i = \sqrt{\frac{\omega^2}{c^2} + j\omega\mu\sigma} = \frac{\omega}{c}\sqrt{1 + j c^2 \frac{\mu\sigma}{\omega}}$$

$$= \frac{\omega}{c}\sqrt{1 + j\frac{1}{\cancel{\mu}\varepsilon}\frac{\cancel{\mu}\sigma}{\omega}} = \frac{\omega}{c}\sqrt{1 + j\frac{\sigma}{\varepsilon\omega}}$$

$$\approx \frac{\omega}{c}\left(1 + j\frac{\sigma}{2\varepsilon\omega}\right) = \frac{\omega}{c} + j\frac{\sigma}{2\varepsilon c} = \frac{\omega}{c} + j\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

for $\omega \gg \frac{\sigma}{\varepsilon}$

$$\Rightarrow \xi = \frac{1}{k_i} = \frac{2\varepsilon c}{\sigma} = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}} \quad \text{for} \quad \omega \gg \frac{\sigma}{\varepsilon}$$

Biological tissue:

$$\varepsilon = \varepsilon_r \cdot \varepsilon_0 = \underset{\substack{\text{(WATER)} \\ \text{@ 1GHz}}}{80\ \varepsilon_0} = 80 \cdot 8.8\ 10^{-12}\ \frac{F}{m}$$

$$\mu = \mu_r \cdot \mu_0 = 1\ \mu_0 = 4\pi\ 10^{-7}\ \frac{H}{m}$$

$$c = \frac{c_0}{\sqrt{80}} = \frac{3\ 10^8}{\sqrt{80}} = .34\ 10^8\ \frac{m}{s}$$

$$\sigma = \underset{\text{(BRAIN)}}{0.5\ \frac{1}{\Omega m}}$$

$$\frac{\sigma}{\varepsilon} = \frac{0.5}{80 \cdot 8.8\ 10^{-12}}\ \frac{\frac{1}{\Omega m}}{\frac{F}{m}} = .710\ 10^9\ \frac{1}{s}$$

$$\Rightarrow \xi = 2c \Big/ \frac{\sigma}{\varepsilon} = \frac{2 \cdot .34\ 10^8\ \frac{m}{s}}{.710\ 10^9\ \frac{1}{s}} = \underline{9.6\ cm} \quad \text{for}$$

$$f = \frac{\omega}{2\pi} \gg \frac{.710\ 10^9}{2\pi}\ Hz = \underline{110\ MHz}$$

absorps $(1 - e^{-2})$ of RF power $= 86\%$

Wave decay:

$$\underline{\text{POWER}} = \varepsilon \cdot \frac{|\vec{E}|^2}{2} = \varepsilon \frac{|\vec{E}_0|^2}{2} \cdot e^{-2k_i \cdot r}$$

$= \vec{r} \cdot \hat{\vec{k}}$

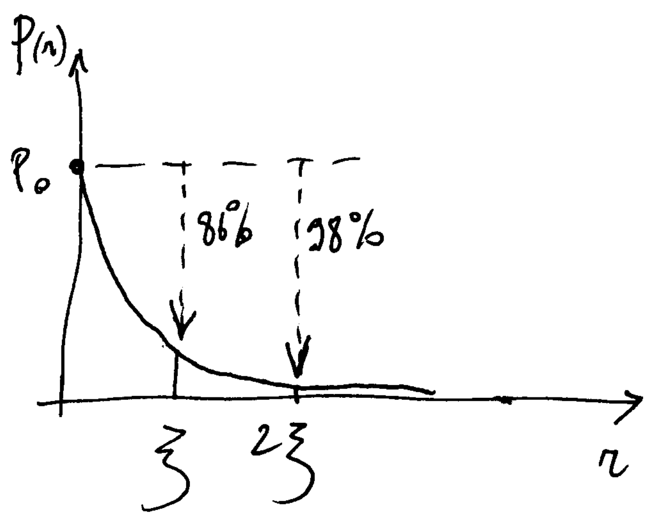
$$\Rightarrow P(r) = P(0) \cdot e^{-2\frac{r}{\xi}}$$

$\Rightarrow$ Power = reduced by $1 - e^{-2} = 86\%$

over a distance $r = \xi$

and reduced by $1 - e^{-4} = 98\%$

over a distance $r = 2\xi$



Cell Phone Radiation and Head Tissue Absorption

A typical cell phone radiates a peak power of $P_0 = 1\ W$. The radiation pattern is not uniform, and in the far-field limit has a doughnut-shaped angular directivity profile with radiation power aerial density $[W/m^2]$:

$$U(r, \varphi) = c\,\frac{\sin^2 \varphi}{r^2}$$

where r is the distance from the cell phone, and φ is the angle relative to the cell antenna axis.

- Show that the net radiated power flux through any sphere with radius r from the cell phone is independent of distance r. Write the constant c in terms of the net radiated power P_0.
- Now consider the absorption due to tissue volume conduction as studied in the previous problem. An unsuspecting user starts a call, holding the cell phone flat on the ear and exposing the head to a dose of RF EM radiation. Modeling the head as a sphere with diameter $D = 15\ cm$, with the antenna axis tangential to the head perimeter ($\varphi = \pi/2$ at the head center), find an estimate of the total RF power absorbed in the head tissue. You may neglect near-field effects, as well as the effect of diffraction by the head dielectrics on the radiation profile, to arrive at a simple but reasonable estimate. Would you expect that this approximation leads to an underestimate or an overestimate of the total absorbed power, and why?
- Now reestimate the total absorbed power in the case where the cell phone is held at a right angle, with the antenna axis perpendicular to the head perimeter ($\varphi = 0$ at the head center). Does the angle at which the cell phone is held make a big difference, and why?

Cell phone: $\quad U(r,\varphi) = c \dfrac{\sin^2\varphi}{r^2}$

$$P_0 = \text{Total power} = \oiint_{S(r)} U(\vec{r})\, dS = \int_0^{2\pi}\int_0^{\pi} U(r,\varphi) r^2 d\theta \sin\varphi\, d\varphi$$

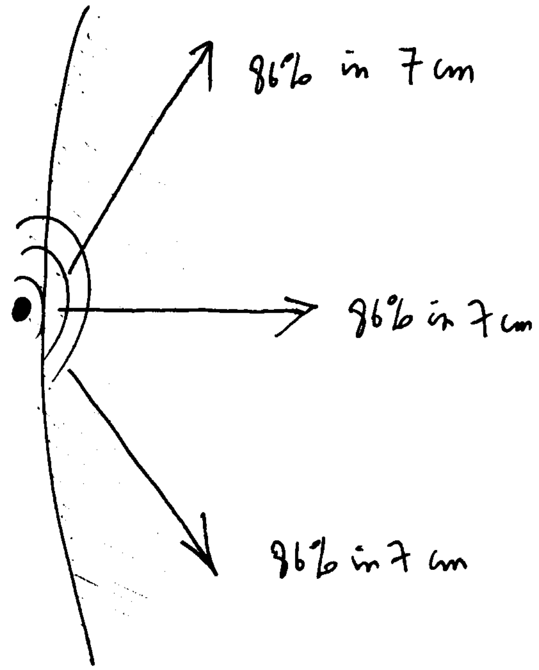
$$= 2\pi \cancel{r^2} \frac{c}{\cancel{r^2}} \int_0^{\pi} \underbrace{\sin^2\varphi}_{\substack{(1-\cos^2\varphi)\\ =(1-x^2)}} \cdot \underbrace{\sin\varphi\, d\varphi}_{-d(\cos\varphi) = -dx}$$

$$= 2\pi c \int_{+1}^{-1} (1-x^2)(-dx) = 2\pi c \int_{-1}^{+1} (1-x^2)\, dx$$

$$= 2\pi c \left[x - \frac{x^3}{3}\right]_{-1}^{+1} = 2\pi c\, 2\left(1-\frac{1}{3}\right) = \frac{8\pi}{3} c$$

$$\Rightarrow P_0 = \frac{8\pi}{3} c \quad \text{or} \quad c = \frac{3}{8\pi} P_0$$

Absorbed power : $\approx \frac{1}{2} P_0$ independent of angle!



independent of radiation pattern, as long as it is symmetric

e.g.