BENG 221: Mathematical Methods in Bioengineering

Lecture 18 Supplement

Magnetostatics, Electrodynamics and Maxwell's Equations

References

http://farside.ph.utexas.edu/teaching/em/lectures/lectures.html

http://www.numericana.com/answer/maxwell.htm

http://en.wikipedia.org/wiki/Magnetostatics
http://en.wikipedia.org/wiki/Biot-Savart_law
http://en.wikipedia.org/wiki/Magnetic_potential
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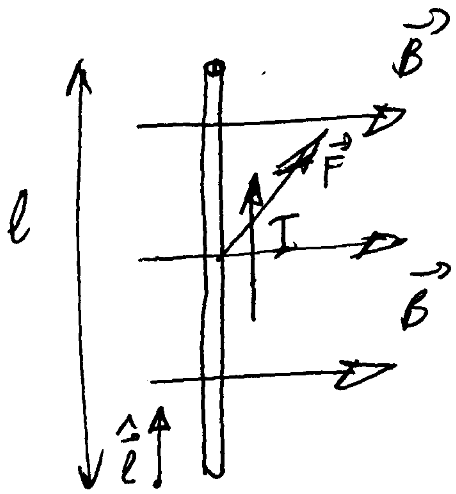
http://en.wikipedia.org/wiki/Faraday%27s_law_of_induction
http://en.wikipedia.org/wiki/Amp%C3%A8re%27s_circuital_law
http://en.wikipedia.org/wiki/Maxwell%27s_equations
http://en.wikipedia.org/wiki/Electromagnetic_radiation

MAGNETIC FIELD

$$\underset{\text{FORCE}}{\vec{F}} = \underset{\text{CHARGE}}{q} \cdot (\underset{\text{VELOCITY}}{\vec{v}} \times \underset{\text{MAGNETIC FIELD}}{\vec{B}})$$

$$[N] = [C \, \tfrac{m}{s} \, T] \quad \text{or} \quad [T] = \left[\frac{N}{m.A}\right]$$

For instance, force on wire carrying a current:



$$\vec{F} = I \vec{\ell} \times \vec{B}$$

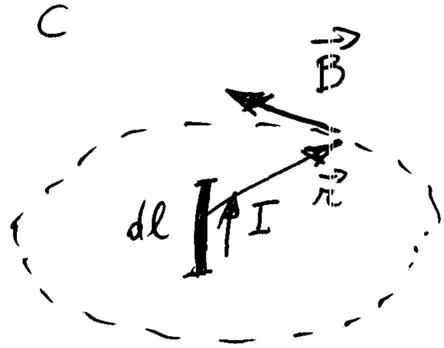
Net electrostatic and magnetostatic force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \text{Lorentz Force}$$

BIOT- SAVART

Magnetic field caused by static currents:

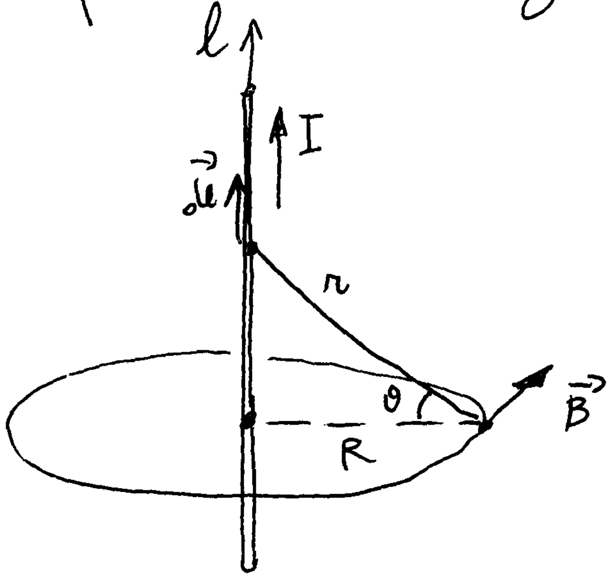
- loop: $\vec{B} = \frac{\mu_0}{4\pi} \int_C \frac{I\, d\vec{l} \times \hat{\vec{r}}}{r^2}$ for current I [A]



$\mu_0 = 4\pi 10^{-7}\ \frac{H}{m}$ or $\frac{N}{A^2}$

- continuum: $\vec{B} = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{J} \times \hat{\vec{r}}}{r^2}\, dV$ for current density $\vec{J}$ $\left[\frac{A}{m^2}\right]$

Example: current along an infinite cable/wire:



$$\vec{dl} \times \hat{\vec{r}} = \cos\theta \cdot dl$$

where $r\cos\theta = R$

and $r^2 = R^2 + l^2$

$$|\vec{B}| = \frac{\mu_0 I}{4\pi} \cdot \int_{-\infty}^{+\infty} \frac{\cos\theta\, dl}{r^2} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{\frac{R}{r}\, dl}{r^2}$$

$$= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{+\infty} \frac{R\, dl}{(R^2+l^2)^{3/2}} = \frac{\mu_0 I}{4\pi} \cdot \frac{1}{R} \cdot \underbrace{\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^{3/2}}}_{=2} = \frac{\mu_0}{2\pi} \frac{I}{R}$$

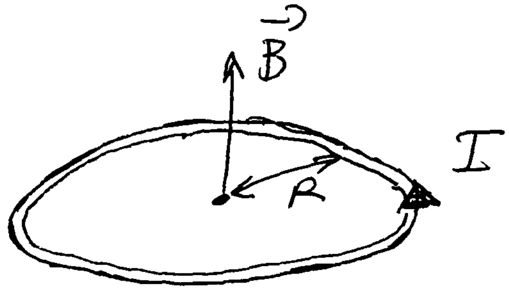
$$\left(\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^{3/2}} = \int_{-\infty}^{+\infty} \frac{\cosh a}{\cosh^3 a}\, da = \int_{-\infty}^{+\infty} \frac{da}{\cosh^2 a} = \left[\tanh a\right]\Big|_{-\infty}^{+\infty} = +1-(-1) = 2 \right)$$

$x = \sinh a$

$1 + x^2 = \cosh^2 a$

$dx = \cosh a \cdot da$

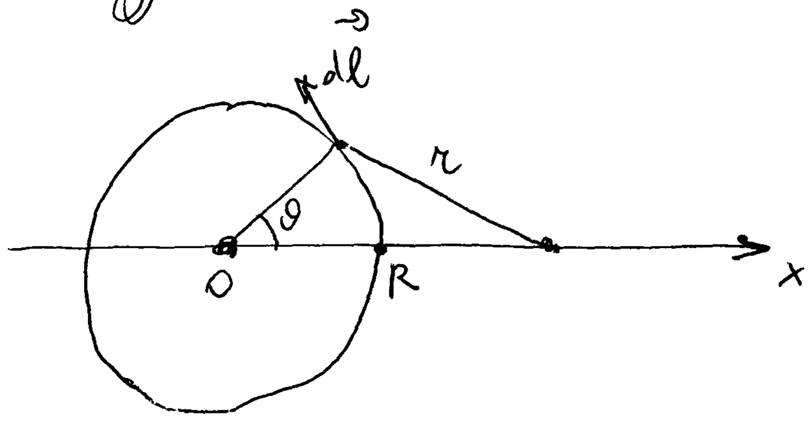
Example: circular loop current:



$\vec{B}$ at center:

$$\frac{\mu_0}{4\pi}\oint \frac{I\,d\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\cdot\frac{I}{R^2}\cdot\underbrace{\oint d\vec{l}\times\hat{r}}_{2\pi R} = \frac{\mu_0 I}{2R}$$

$\vec{B}$ off center



exercise...

Magnetic Vector Potential :

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

with $\vec{A} = \frac{\mu_0}{4\pi} \int_C \frac{I\,\vec{dl}}{r} = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{j}\,dV}{r}$

loop $\qquad$ continuum

because $\vec{\nabla} \times \frac{\vec{dl}}{r} = \vec{\nabla}\left(\frac{1}{r}\right) \times \vec{dl}$

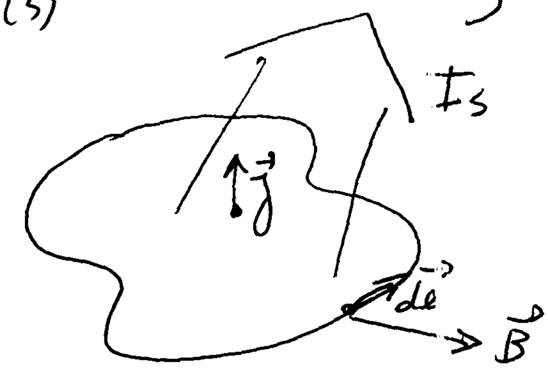
$$= -\frac{\hat{\vec{r}}}{r^2} \times \vec{dl}$$

$(\vec{a} \times \vec{b} = -\vec{b} \times \vec{a})$

$$= \vec{dl} \times \frac{\hat{\vec{r}}}{r^2}$$

Ampère's Law:

- Integral form: $$\oint_{C(S)} \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{j} \cdot \vec{n}\, dS = \mu_0 \cdot I_S$$



- Differential form: $$\vec{\nabla} \times \vec{B} = \mu_0 \cdot \vec{j}$$

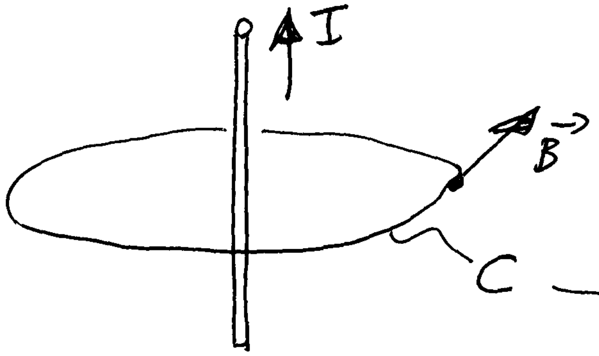
follows from Biot-Savart:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$$

$$= \underbrace{\vec{\nabla} \underbrace{\frac{\mu_0}{4\pi} \int_V \vec{j} \cdot \vec{\nabla}\left(\frac{1}{r}\right) dV}_{\vec{j}}}_{\vec{\nabla} \vec{j} = 0 \text{ (static)}} - \frac{\mu_0}{4\pi} \int \vec{j} \cdot \underbrace{\vec{\nabla}^2 \left(\frac{1}{r}\right)}_{-4\pi\delta(r)} dV$$

$$= \mu_0 \cdot \vec{j}$$

Example: infinite cable/wire revisited:



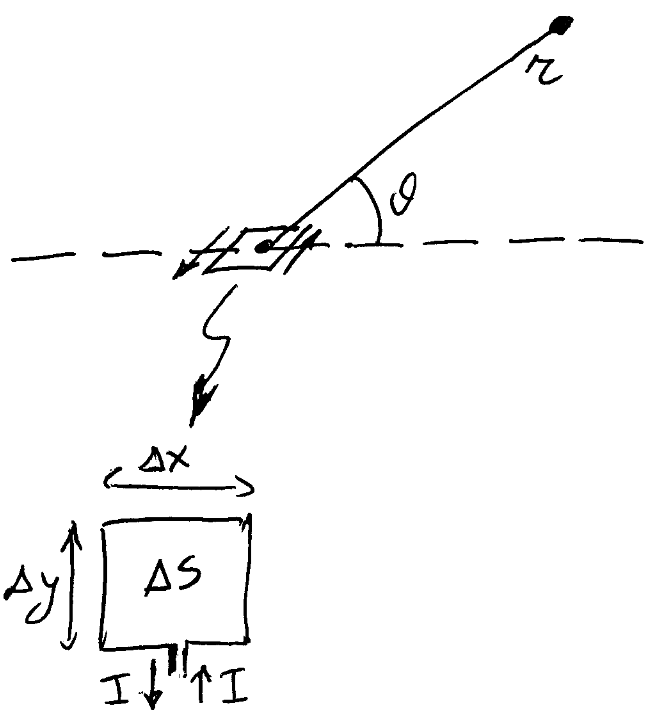
$$\oint_{C(S)} \vec{B} \cdot \vec{dl} = |\vec{B}| \cdot 2\pi R$$

$$\mu_0 \iint_S \vec{J} \cdot \vec{n} dS = \mu_0 \cdot I$$

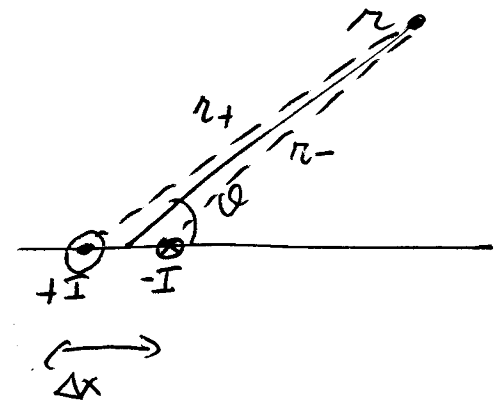
$$\Rightarrow |\vec{B}| \, 2\pi R = \mu_0 \cdot I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi R}$$

Example: magnetic dipole



or



$$|\vec{A}| = \frac{\mu_0}{4\pi} I \left(\frac{\Delta y}{r^+} - \frac{\Delta y}{r^-} \right)$$

with $r^+ \approx r + \cos\vartheta \cdot \frac{\Delta x}{2} = r\left(1 + \cos\vartheta \frac{\Delta x}{2r}\right)$

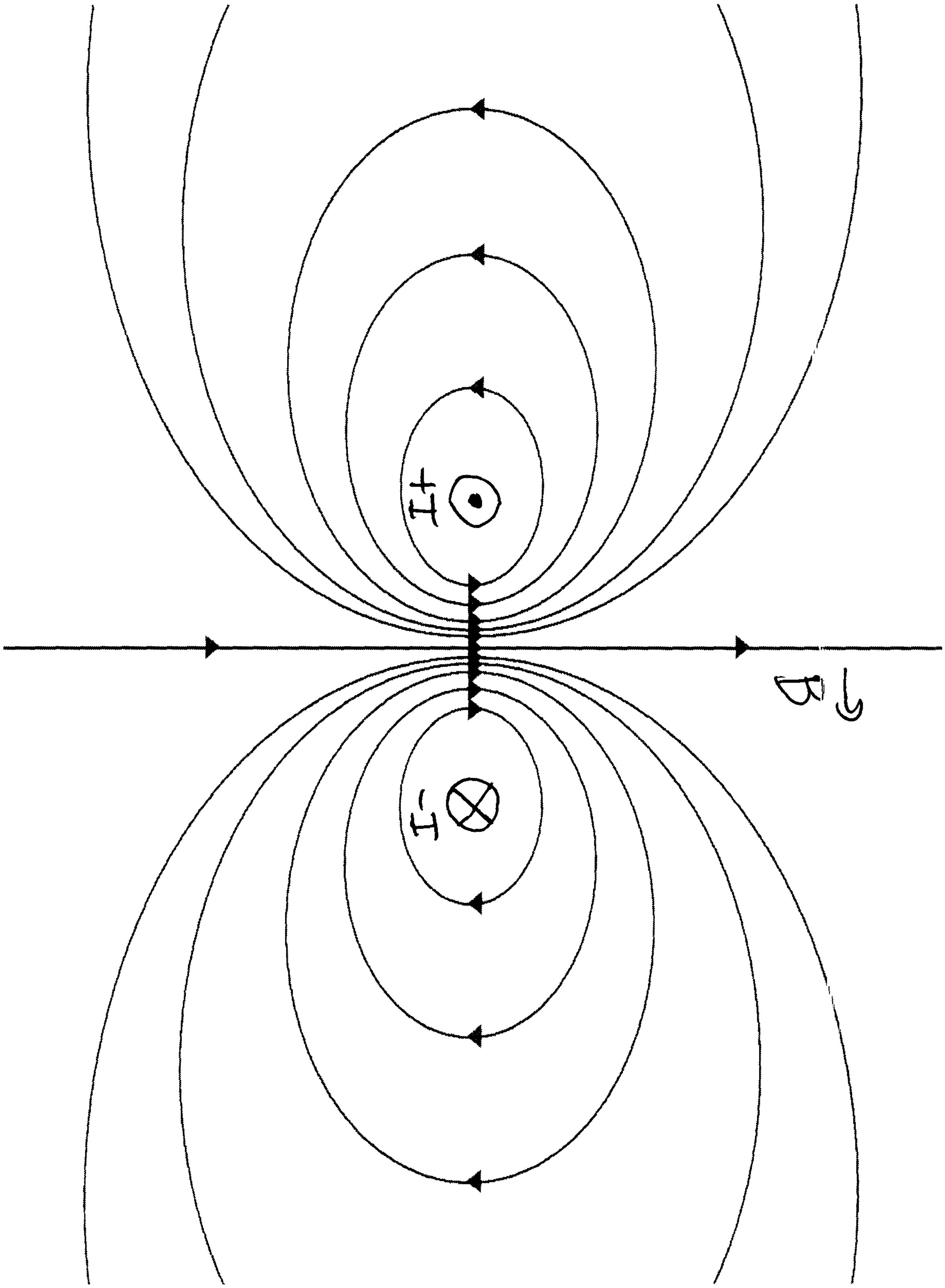
$r^- \approx r - \cos\vartheta \ \frac{\Delta x}{2} = r\left(1 - \cos\vartheta \frac{\Delta x}{2r}\right)$

$$\Delta y \left(\frac{1}{r^+} - \frac{1}{r^-} \right) \approx \underbrace{\Delta x \Delta y}_{\text{area } \Delta S} \ \frac{\mu_0 I}{4\pi} \cos\vartheta \ \frac{1}{r^2}$$

$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \cdot \frac{1}{r^2} \cdot (\vec{m} \times \hat{\vec{r}}) \qquad \vec{m} = I \, \Delta S \, . \, \hat{\vec{z}}$$

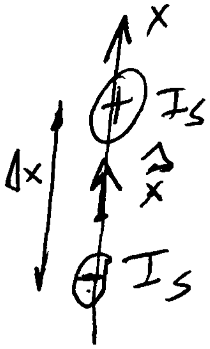
magnetic dipole moment

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left(3(\vec{m} \cdot \hat{\vec{r}}) \hat{\vec{r}} - \vec{m} \right)$$


I+
I-
B

Current dipole magnetic field

e.g.: MEG, MCG, ...



$$\vec{A} = \frac{\mu_0 I_s \cdot \hat{\vec{x}}}{4\pi} \cdot \left(\frac{1}{r^+} - \frac{1}{r^-} \right)$$

$$\approx \frac{\mu_0 I_s \hat{\vec{x}}}{4\pi} \cdot \frac{\Delta x \cos\vartheta}{r^2} = \frac{\mu_0 \hat{\vec{x}}}{4\pi} \cdot \frac{\vec{p} \cdot \hat{\vec{r}}}{r^2}$$

with $\vec{p} = I_s \cdot \Delta x \cdot \hat{\vec{x}}$

e.g.: $I_s = 1\,mA$

$\Delta x = 1\,mm$ $\qquad \mu_0 = 4\pi\, 10^{-7} \frac{H}{m}$

$r = 10\,cm$

$$\rightarrow |\vec{A}| \hat{=} \frac{\mu_0}{4\pi} \cdot I_s \frac{\Delta x}{r^2} \approx 10^{-7} \cdot 10^{-9} \cdot \frac{10^{-3}}{(10^{-})^2} = 10^{-17} \frac{T}{m}$$

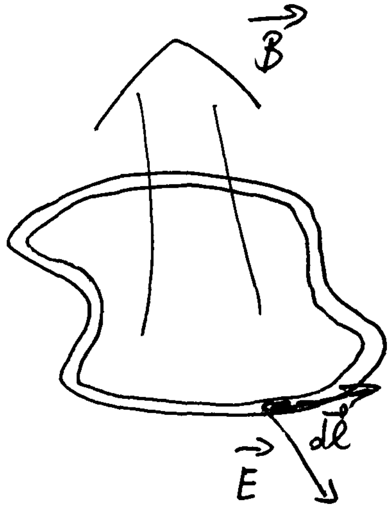
small! – requires superconducting magnetosensors

ELECTRODYNAMICS AND MAXWELL'S EQUATIONS

Faraday's Law of Induction:

- Integral form:

$$\oint_{C(S)} \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \vec{n}\, dS$$



- Differential form:

Stokes $\Rightarrow$ $$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}$$

Can be derived as a consequence of the Lorentz force $\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$ under translation of coordinate frame at linear velocity $\vec{v}$.

So far ... (in vacuum)

$$\left\{\begin{array}{lll} \text{Gauss: } \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} & \text{or} & \oint\!\!\!\oint_{S(V)} \vec{E}\cdot\vec{n}\,dS = \frac{1}{\varepsilon_0}\iiint_V \rho\,dV \\ \text{Faraday: } \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \text{or} & \oint_{C(S)} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot\vec{n}\,dS \end{array}\right.$$

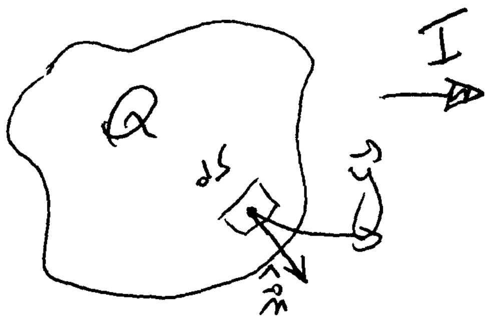
$$\left\{\begin{array}{lll} \text{Gauss for magnetism: } \vec{\nabla}\cdot\vec{B} = 0 & \text{or} & \oint\!\!\!\oint_{S(V)} \vec{B}\cdot\vec{n}\,dS = 0 \\ \text{Ampère: } \vec{\nabla}\times\vec{B} = \mu_0 \vec{j} & \text{or} & \oint_{C(S)} \vec{B}\cdot d\vec{l} = \mu_0 \iint_S \vec{j}\cdot\vec{n}\,dS \end{array}\right.$$

Problem: Ampère's law violates conservation of charge:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 \Rightarrow \vec{\nabla}\,\vec{j} = 0$$

but $\vec{\nabla}\,\vec{j} = 0$ only if there is no charge dynamics. (static case)

Conservation of charge:



$$\frac{dQ}{dt} = -I$$

$\frac{dQ}{dt}$ ↓ CHARGE ACCUMULATION INTERNALLY

I ↓ OUTWARD CURRENT

- Integral notation:

$$\frac{\partial}{\partial t} \iiint_V \rho \, dV = - \oint_{S(V)} \vec{j} \cdot \vec{n} . S$$

- Differential form:

$$\frac{\partial}{\partial t} \rho = - \vec{\nabla} \cdot \vec{j}$$

Maxwell's Correction to Ampère's Law:

$$\vec{\nabla} \cdot \vec{j} = - \frac{\partial}{\partial t} \rho$$

$$\Downarrow$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \left(\vec{\nabla} \cdot \vec{j} + \frac{\partial}{\partial t} \rho \right) = 0$$

$$\Downarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{E} \right)$$

$$\Downarrow$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{E}$$ Differential Form

or;

$$\oint_{C(S)} \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{j} \cdot \vec{n} \, dS + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \iint_S \vec{E} \cdot \vec{n} \, dS$$

Integral Form

In Summary:

Gauss: $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ or: $\displaystyle\oint\!\!\oint_{S(V)} \vec{E}\cdot\vec{n}\,dS = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$

Faraday: $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ or: $\displaystyle\oint_{C(S)} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot\vec{n}\,dS$

Gauss for magnetism: $\vec{\nabla}\cdot\vec{B} = 0$ or: $\displaystyle\oint\!\!\oint_{S(V)} \vec{B}\cdot\vec{n}\,dS = 0$

Ampère with Maxwell correction: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial}{\partial t}\vec{E}$ or: $\displaystyle\oint_{C(S)} \vec{B}\cdot d\vec{l} = \mu_0\iint_S \vec{J}\cdot\vec{n}\,dS + \mu_0\varepsilon_0\frac{\partial}{\partial t}\iint_S \vec{E}\cdot\vec{n}\,dS$

In dielectric media: $\underbrace{\varepsilon_0}_{\text{VACUUM PERMITTIVITY}} \rightarrow \varepsilon = \varepsilon_0\cdot\underbrace{\varepsilon_r}_{\text{RELATIVE PERMITTIVITY}} = \varepsilon_0(1+\chi)$

In magnetic susceptible media: $\underbrace{\mu_0}_{\text{VACUUM SUSCEPTIBILITY}} \rightarrow \mu = \mu_0\cdot\underbrace{\mu_r}_{\text{RELATIVE SUSCEPTIBILITY}} = \mu_0(1+\chi_m)$