BENG 221: Mathematical Methods in Bioengineering

Lecture 19

Wave Equation in One Dimension: Vibrating Strings and Pressure Waves

References

Haberman APDE, Ch. 4 and Ch. 12.

http://en.wikipedia.org/wiki/Wave_equation

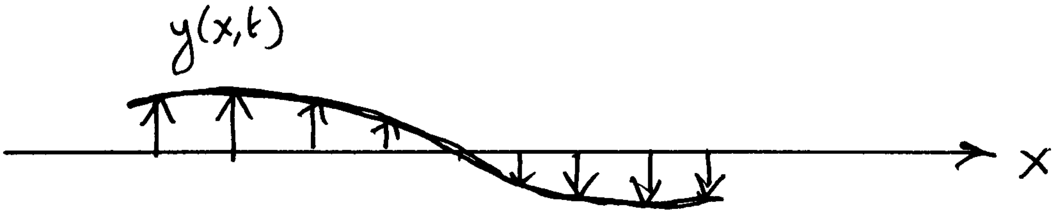
http://en.wikipedia.org/wiki/Vibrating_string

http://en.wikipedia.org/wiki/Longitudinal_wave

TRANSVERSAL WAVES IN SOLIDS

e.g. VIBRATING STRING IN 1-D

For incompressible, elastic solids, displacement is primarily TRANSVERSAL, or PERPENDICULAR to the longitudinal axis of wave propagation:

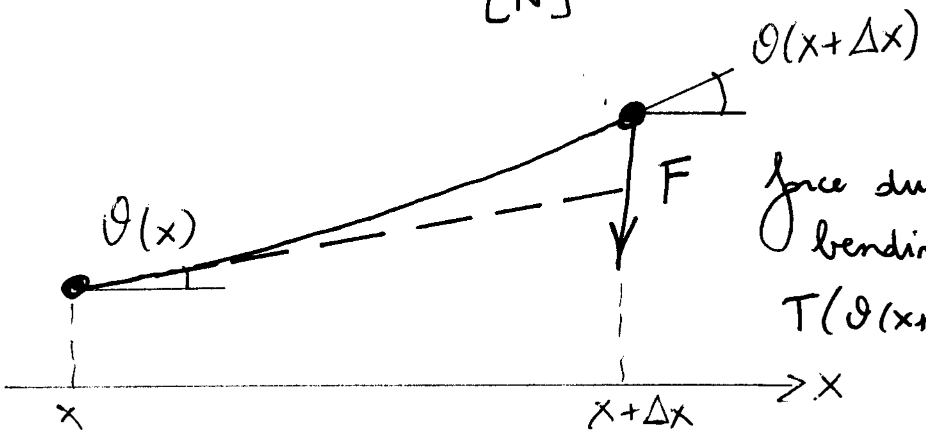


Newton:

MASS × ACCELERATION = FORCE

$$\rho_L \cdot \Delta x \quad \cdot \quad \frac{\partial^2 y}{\partial t^2} \quad = \quad T \quad \cdot \left(\vartheta(x+\Delta x) - \vartheta(x) \right)$$

- ρ_L: LINEAR MASS DENSITY $\left[\frac{kg}{m}\right]$
- Δx: $[m]$
- $\frac{\partial^2 y}{\partial t^2}$: $\left[\frac{m}{s^2}\right]$
- T: TENSION (STIFFNESS) $[N]$



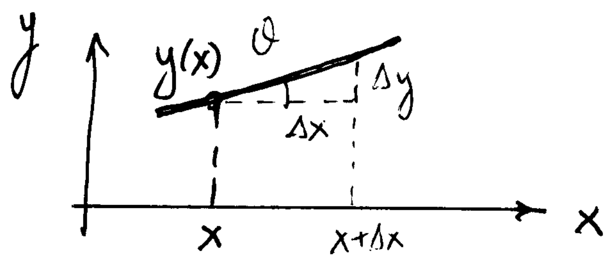
force due to bending = $T(\vartheta(x+\Delta x) - \vartheta(x))$

In the limit $\Delta x \to 0$:

$$\vartheta(x+\Delta x) - \vartheta(x) \approx \frac{\partial \vartheta}{\partial x} \cdot \Delta x$$

Also, in the limit $\vartheta \ll 1$ (small displacements y):

$$\vartheta(x) = \text{Arctan}\left(\frac{\partial y}{\partial x}\right) \approx \frac{\partial y}{\partial x}$$



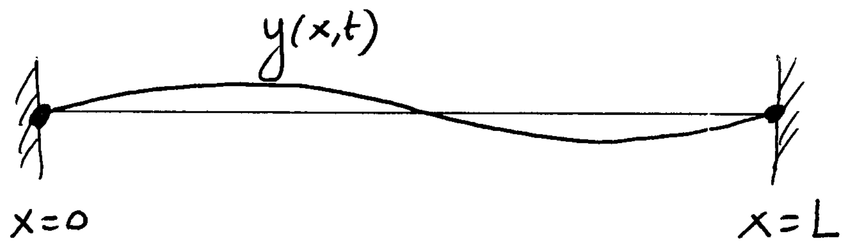
$$\Rightarrow \quad \rho_L \, \Delta x \cdot \frac{\partial^2 y}{\partial t^2} = T \, \frac{\partial^2 y}{\partial x^2} \cdot \Delta x$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = c^2 \, \frac{\partial^2 y}{\partial x^2}$$ WAVE EQUATION IN 1-D

where $c = \sqrt{\dfrac{T}{\rho_L}}$ is the WAVE VELOCITY

$$\left[\sqrt{\frac{N}{\frac{kg}{m}}}\right] = \left[\sqrt{\frac{\cancel{kg}\,\frac{m}{s^2}}{\frac{\cancel{kg}}{m}}}\right] = \left[\frac{m}{s}\right]$$ OK!

e.g. Vibrating string:



$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + f(x,t) \quad \text{with:} \quad \begin{cases} \text{I.C:} & y(x,0) = g(x) \\ & \dot{y}(x,0) = h(x) \\ \text{B.C:} & y(0,t) = y_0(t) \\ \text{(VALUE)} & y(L,t) = y_L(t) \end{cases}$$

$f(x,t)$ ↓ FORCE per unit mass impinging on the string

Homogeneous case: $f(x,t) \equiv 0$; $y_0 \equiv 0$; $y_L \equiv 0$

Separation of variables: $y(x,t) = \phi(x) \cdot G(t)$

$$\frac{\frac{d^2\phi}{dx^2}}{\phi} = \frac{1}{c^2} \cdot \frac{\frac{d^2 G}{dt^2}}{G} = -\lambda$$

$(\lambda \geq 0)$

<u>TIME</u>: $G(t) = A \cos(\sqrt{\lambda} ct) + B \sin(\sqrt{\lambda} ct)$

<u>SPACE</u>: $\phi(x) = C \cos(\sqrt{\lambda} x) + D \sin(\sqrt{\lambda} x)$

B.C. : $C = 0$; $\sqrt{\lambda} \cdot L = n \cdot \pi$, $n = 1, 2, \ldots \infty$

$$\Rightarrow \quad y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

The string resonates with frequencies $\omega_n = \frac{n\pi c}{L}$, or

$$f_n = \frac{\omega_n}{2\pi} = n\frac{c}{2L} = n\,\frac{1}{2L}\sqrt{\frac{T}{\rho_L}},$$

harmonics of the fundamental frequency $\frac{1}{2L}\sqrt{\frac{T}{\rho_L}}$

- lower pitch for longer, heavier strings
- higher pitch for higher tension (tuning)

I.C. : – DISPLACEMENT y ("plucking" the string):

$$y(x,0) = g(x) \Rightarrow \begin{cases} A_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ B_n = 0 \end{cases}$$

– MOMENTUM $\dot{y} = \frac{\partial y}{\partial t}$ ("hitting the string"):

$$\dot{y}(x,0) = h(x) \Rightarrow \begin{cases} A_n = 0 \\ B_n = \frac{2}{n\pi c}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$$

Inhomogeneous case: $f(x,t) \neq 0$ or $y_0(t) \neq 0$ or $y_L(t) \neq 0$
or $\dot{y}_0(t) \neq 0$ or $\dot{y}_L(t) \neq 0$

→ Green's functions for DISPLACEMENT and MOMENTUM

Green's function for DISPLACEMENT $y(x_0, t_0)$:

$$G_D(x,t;x_0,t_0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x_0}{L}\right)\cos\left(\frac{n\pi c(t-t_0)}{L}\right)$$

IMPULSE DISPLACEMENT @ x_0, t_0 : $y(x,t) = \delta(x-x_0)\,\delta(t-t_0)$

Green's function for MOMENTUM $\frac{\partial y}{\partial t}(x_0, t_0)$:

$$G_M(x,t;x_0,t_0) = \sum_{n=1}^{\infty} \frac{2}{n\pi c}\cdot\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi c(t-t_0)}{L}\right)$$

IMPULSE MOMENTUM @ x_0, t_0 : $\dot{y}(x,t) = \delta(x-x_0)\,\delta(t-t_0)$

Example: solution for $f(x_0,t_0)$ with zero I.C. & B.C. :

f ↓ FORCE per unit mass

$\Rightarrow$ source term for MOMENTUM

$$\Rightarrow y(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi c}\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot\int_0^L\int_0^t f(x_0,t_0)\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi c(t-t_0)}{L}\right)\cdot dx_0\,dt_0$$

NOTE: Homogeneous solution can also be written as a sum of FORWARD and REVERSE propagating waves:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

$$\left\{ \begin{array}{l} \sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right) \\ \sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right) \end{array} \right. \Rightarrow$$

$$y(x,t) = \underbrace{y_F(x-ct)}_{\text{FORWARD WAVE}} + \underbrace{y_R(x+ct)}_{\text{REVERSE WAVE}}$$

with

$$y_F(x-ct) = \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\left(\frac{n\pi(x-ct)}{L}\right) + \frac{B_n}{2}\cos\left(\frac{n\pi(x-ct)}{L}\right)\right)$$

$$y_R(x+ct) = \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\left(\frac{n\pi(x+ct)}{L}\right) - \frac{B_n}{2}\cos\left(\frac{n\pi(x+ct)}{L}\right)\right)$$

Indeed, any solution of the form $\Psi(x \pm ct)$ is a solution to the 1-D homogeneous wave equation ("method of characteristics").

METHOD OF CHARACTERISTICS : 1-D wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{with} \quad \text{B.C:} \begin{cases} y(0,t) = 0 \\ y(L,t) = 0 \end{cases} \quad \text{and I.C.:} \begin{cases} y(x,0) = g(x) \\ \dot{y}(x,0) = h(x) \end{cases}$$

Characteristic solution :

$$y(x,t) = \underbrace{y_F(x-ct)}_{\text{FORWARD WAVE}} + \underbrace{y_R(x+ct)}_{\text{REVERSE WAVE}}$$

I.C. :

$$\begin{cases} y(x,0) = y_F(x) + y_R(x) = g(x) \\ \dot{y}(x,0) = -c\,\dot{y}_F(x) + c\,\dot{y}_R(x) = h(x) \end{cases}$$

$$\Downarrow$$

$$-y_F(x) + y_R(x) = \frac{1}{c}\int_{-\infty}^{x} h(x_0)\,dx_0$$

$$\Rightarrow \begin{cases} y_F(x) = \frac{1}{2} g(x) - \frac{1}{2c}\int_{-\infty}^{x} h(x_0)\,dx_0 \\ y_R(x) = \frac{1}{2} g(x) + \frac{1}{2c}\int_{-\infty}^{x} h(x_0)\,dx_0 \end{cases}$$

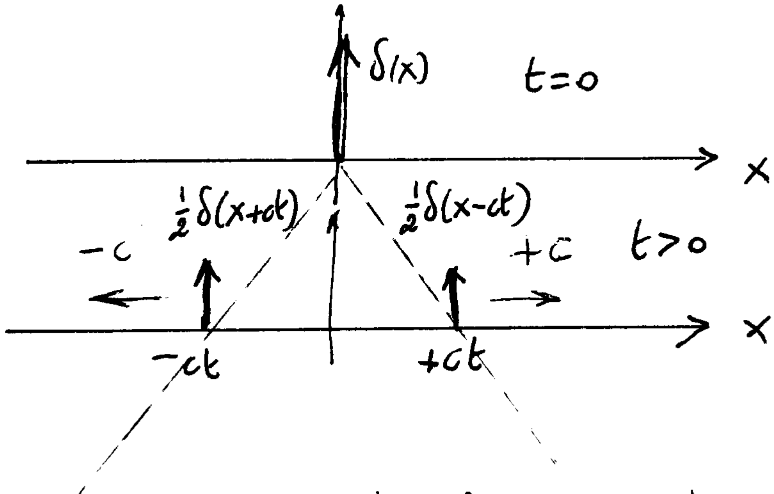
$$\Rightarrow \quad y(x,t) = \frac{1}{2}\left(g(x-ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x_0)\,dx_0$$

NOTE : Account for REFLECTIONS on the BOUNDARIES !

e.g.: No boundary conditions; infinite string:

- DISPLACEMENT Impulse @ center $x=0$:

$$\begin{cases} g(x) = \delta(x) \\ h(x) = 0 \end{cases} \Rightarrow \quad y(x,t) = \frac{1}{2}\left(\delta(x-ct) + \delta(x+ct)\right)$$

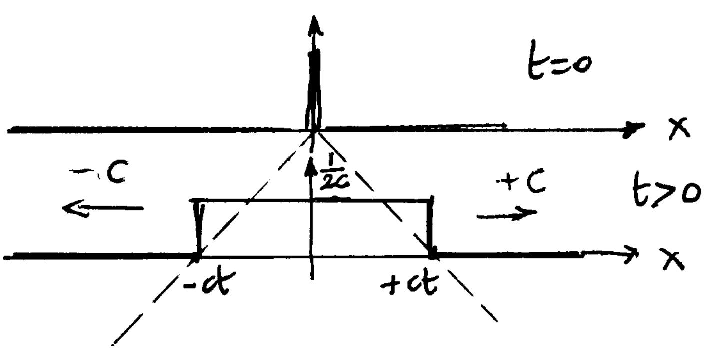


→ Impulse separates in two halves, each traveling at c in opposing directions.

- MOMENTUM Impulse @ center $x=0$:

$$\begin{cases} g(x) = 0 \\ h(x) = \delta(x) \end{cases} \Rightarrow \quad y(x,t) = \frac{1}{2c}\left(H(x+ct) - H(x-ct)\right)$$

$$= \begin{cases} \frac{1}{2c} & \text{for } -ct \leq x \leq ct \\ 0 & \text{otherwise} \end{cases}$$

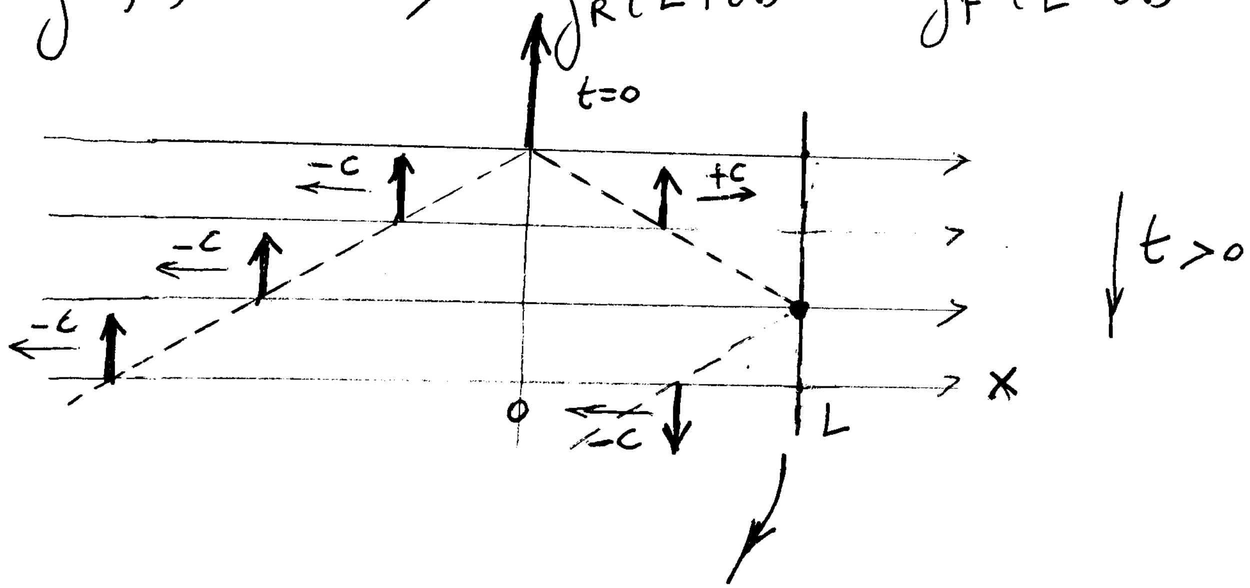


→ Unity amplitude wavefront propagates at velocity c in opposing directions!

e.g.: Semi-infinite domain; single boundary (ZERO VALUE):

- DISPLACEMENT Impulse @ $x=0$; ZERO VALUE @ $x=L$:

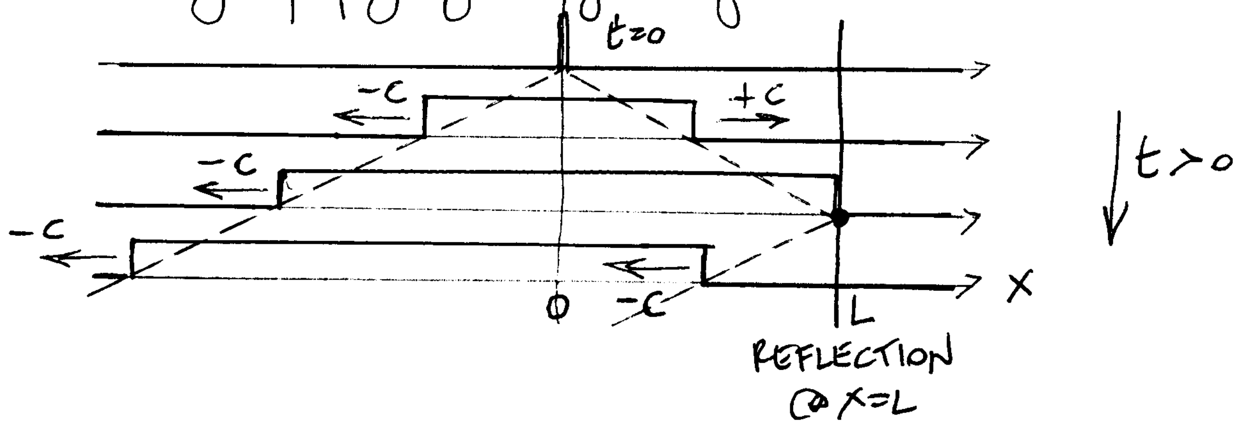
$$y(L,t) = 0 \Rightarrow y_R(L+ct) = -y_F(L-ct)$$



REFLECTION @ $x=L$: wave reverses polarity AND direction

- MOMENTUM Impulse @ $x=0$; ZERO VALUE @ $x=L$:

same for propagating unity wavefront:

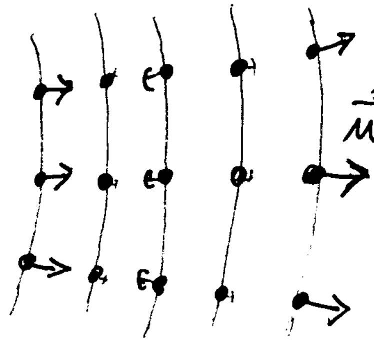


- ZERO FLUX @ $x=L$: SAME, except the wave RETAINS polarity when reverting direction.

LONGITUDINAL WAVES IN GASES & LIQUIDS

e.g. (ULTRA) SOUND in 3-D

In COMPRESSIBLE media such as blood gases and, to some extent, TISSUE, displacement is primarily LONGITUDINAL along the axis of wave propagation:



$\vec{u}(\vec{r},t)$ DISPLACEMENT VECTOR $[m]$

$p(\vec{r},t)$ PRESSURE $\left[\frac{N}{m^2} = \frac{kg}{m s^2}\right]$

Newton:

MASS × ACCELERATION = FORCE

$$\rho \cdot \frac{\partial^2 \vec{u}}{\partial t^2} = -\vec{\nabla} p$$

VOLUME MASS DENSITY $\left[\frac{kg}{m^3}\right]$

$\left[\frac{m}{s^2}\right]$

$\left[\frac{1}{m}\right]\left[\frac{N}{m^2}\right]$

FORCE DENSITY = − GRADIENT OF PRESSURE

(pressure = mechanical energy of the compressed medium)

Bulk modulus B (or compressibility $k = \frac{1}{B}$):

$$B = - p \Big/ \frac{\Delta V}{V} \qquad \text{units: } \left[\frac{N}{m^2}\right] = \left[\frac{kg}{m s^2}\right]$$

pressure change per unit RELATIVE volume change

Volume change ΔV due to compression of a small volume V:

$$\Delta V = \oint_{S(V)} \vec{u} \cdot \vec{n} \, dS = \int_V \vec{\nabla} \cdot \vec{u} \, dV \to \vec{\nabla} \cdot \vec{u} \cdot V$$

or $$\frac{\Delta V}{V} = \vec{\nabla} \cdot \vec{u}$$

$$\Rightarrow p = -B \, \vec{\nabla} \cdot \vec{u} = -B \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)$$

$$\Rightarrow \frac{\partial^2 p}{\partial t^2} = -B \, \vec{\nabla} \cdot \left(\frac{\partial^2 \vec{u}}{\partial t^2} \right) = -B \, \vec{\nabla} \cdot \left(-\frac{1}{\rho} \vec{\nabla} p \right)$$

$$= c^2 \, \vec{\nabla} \cdot \vec{\nabla} \, p = c^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right)$$

with WAVE VELOCITY $c = \sqrt{\frac{B}{\rho}}$

NOTES:

- All media are, to some extent, COMPRESSIBLE (even solids).

  → LONGITUDINAL WAVES with: $c = \sqrt{\frac{B}{\rho}}$ → B: BULK MODULUS; ρ: VOLUME MASS DENSITY

  → TRANSVERSAL WAVES with: $c = \sqrt{\frac{G}{\rho}}$ → G: SHEAR MODULUS

- For a string / cable, propagation and displacement are 1-D

  with: $G \rightarrow T/A$

  $\rho \rightarrow \rho_L/A$

A: CROSS-SECTION AREA

- For a perfect gas:

  $$p \cdot V = RT \Rightarrow \frac{\Delta V}{V} = -\frac{p}{p_0}$$

  p ← change in pressure (volume)

  p_0 ← static pressure (volume)

  $$\Rightarrow B = p_0$$

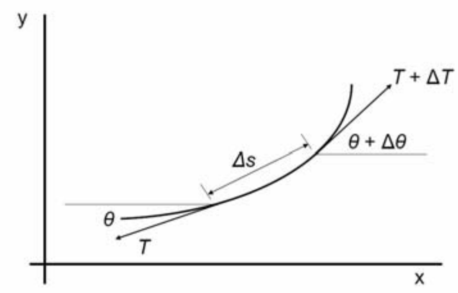
BENG 221
Lecture 17
M. Intaglietta

The one dimensional wave equation. The vibrating string as a boundary value problem

Given a string stretched along the *x* axis, the vibrating string is a problem where forces are exerted in the *x* and *y* directions, resulting in motion in the *x*-*y* plane, when the string is displaced from its equilibrium position within the x-y plane, and then released.

The free body diagram of an element of string of length Δs subjected to a tension *T* is shown. The string material has density ρ. The equation of motion is obtained by applying Newton's second law of motion to the element of length Δs in both directions. For the *x* direction (and ignoring the torsional effects due to the applied torque):

$$(T+\Delta T)\cos(\theta+\Delta\theta)-T\cos\theta=\rho A\Delta s\overline{\frac{\partial^2 x}{\partial t^2}}$$



and in the *y* direction:

$$(T+\Delta T)\sin(\theta+\Delta\theta)-T\sin\theta=\rho A\Delta s\overline{\frac{\partial^2 y}{\partial t^2}}$$

Where the bar over the partial derivative signifies the average acceleration over the element Δs. *A* is the cross section of the string, assumed constant and equal to 1.

Dividing through by Δs and taking the limit $\Delta s \to 0$ we obtain:

$$\frac{\partial}{\partial s}(T\cos\theta)=\rho\frac{\partial^2 x}{\partial t^2}$$
$$\frac{\partial}{\partial s}(T\sin\theta)=\rho\frac{\partial^2 y}{\partial t^2} \qquad (15)$$

since $\cos\theta = \frac{\partial x}{\partial s}$ and $\sin\theta = \frac{\partial y}{\partial s}$ the above equations can be reduced to the form (for constant *T*)

$$T\frac{\partial^2 x}{\partial s^2} = \rho\frac{\partial^2 x}{\partial t^2}$$
$$T\frac{\partial^2 y}{\partial s^2} = \rho\frac{\partial^2 y}{\partial t^2} \qquad (15a)$$

Since there is no motion in the *x* direction $\frac{\partial^2 x}{\partial t^2} = 0$.

Note that $\tan\theta = \frac{\partial y}{\partial x}$, therefore

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \approx \frac{\partial y}{\partial x}$$

for small θ therefore in (15a):

$$T\frac{\partial}{\partial s}\frac{\partial y}{\partial x} = T\frac{\partial}{\partial x}\frac{\partial y}{\partial s} = T\frac{\partial}{\partial x}\sin\theta \approx T\frac{\partial}{\partial x}\frac{\partial y}{\partial x} = T\frac{\partial^2 y}{\partial x^2}$$

and the system of equations reduces to:

$$\frac{\partial^2 x}{\partial t^2} = 0$$
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} \qquad (16)$$

The first equation has a trivial solution. The second equation can be solved by the method of separation of variables by assuming that $y(x,t) = X(x)F(t)$ which leads to:

$X\frac{d^2F}{dt^2} = \frac{T}{\rho}F\frac{d^2X}{dx^2}$ and dividing both sides by $\frac{T}{\rho}XF$ we obtain:

$$\frac{\rho}{T}\frac{1}{F}\frac{d^2F}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -\kappa^2$$

according to previous reasoning indicating that an equality between functions of different variables implies that the functions are equal to a constant. Therefore we can write:

$$\frac{d^2F}{dt^2} = -\kappa^2 \frac{T}{\rho} F \quad and \quad \frac{d^2X}{dt^2} = -\kappa^2 X$$

as previously, the solutions of these equations correspond to the case where the constant is positive and therefore the characteristic equation of the second order differential equation has imaginary coefficients, leading to the following solutions in terms of trigonometric functions:

$$F = A_1 \cos\kappa\sqrt{\frac{T}{\rho}}t + B_1 \sin\kappa\sqrt{\frac{T}{\rho}}t$$
$$X = A_2 \cos\kappa x + B_2 \sin\kappa x$$

Therefore the system of equations given in (16) has product solutions of the form:

$$y(t,x) = \left(A_1 \cos\kappa\sqrt{\frac{T}{\rho}}t + B_1 \sin\kappa\sqrt{\frac{T}{\rho}}t \right)\left(A_2 \cos\kappa x + B_2 \sin\kappa x \right)$$

When $\kappa = 0$ then:

$$F_0 = A_0 + B_0 t$$
$$X_0 = C_0 + D_0 x$$
$$y_0 = \left(A_0 + B_0 t \right)\left(C_0 + D_0 x \right)$$

<u>Boundary conditions. Vibrating string clamped at both ends</u>

If we impose the B.C.s that the string is clamped at the ends, namely $y = 0$ at $x = 0, L$ then there is no motion in the y direction and:

$$y(t,x) = \left(A_1 \cos\kappa\sqrt{\frac{T}{\rho}}t + B_1 \sin\kappa\sqrt{\frac{T}{\rho}}t \right) A_2 = 0$$
$$y_0 = (A_0 + B_0 t)C_0 = 0$$
$$y(t,x) = \left(A_1 \cos\kappa\sqrt{\frac{T}{\rho}}t + B_1 \sin\kappa\sqrt{\frac{T}{\rho}}t \right)\left(A_2 \cos\kappa L + B_2 \sin\kappa L \right) = 0$$
$$y_0 = \left(A_0 + B_0 t \right)\left(C_0 + D_0 L \right) = 0$$

These B.C.s are satisfied by $A_2 = C_0 = D_0 = 0$ and the eigenvalues

$$\kappa L = n\pi \quad n = 1, 2, 3, \ldots$$

leading to eigenfunctions:

$$y_n = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi}{L}\sqrt{\frac{T}{\rho}}t + B_n \sin\frac{n\pi}{L}\sqrt{\frac{T}{\rho}}t \right) \tag{17}$$

where the constant A_1 is now include in A_n and B_n**.** Note that for any time t_0 the string has a configuration that depend son on *n*:

$$y_n(x,t_0) = C_n(t_0)\sin\frac{n\pi x}{L}$$

which describes a family of modes of the string for the specific B.C.s of clamped ends. These are called the **normal modes** of vibration. The intensity of sound depends on the amplitude $C_n = \sqrt{A_n^2 + B_n^2}$ which is derived from:

$$A\cos\theta + B\sin\theta = \sqrt{A^2 + B^2}\,\sin(\theta + \lambda) \quad \lambda = \tan^{-1}\frac{A}{B}$$

The number of oscillations per unit time or **frequency** in cycles per second is:

$$\omega = \frac{1}{2\pi}\frac{n\pi}{L}\sqrt{\frac{T}{\rho}} = \frac{n}{2L}\sqrt{\frac{T}{\rho}}$$

Sound is produced by the superposition of natural frequencies $n = 1,2,3,\ldots$. The normal mode is the first harmonic or fundamental *n* = 1. The larger the natural frequency, the higher the pitch. Tuning is accomplished by varying either L, ρ or *T*. *For* vibrating strings the frequencies of the higher harmonics are all integral multiples of the fundamental. Note that sound is produced by strings vibrating in a lateral direction, however it is transmitted by waves of compression and rarefaction in the longitudinal direction (the direction of propagation).

<u>Standing waves and summation of traveling waves</u>

Each standing wave is composed by the summation of two waves traveling in opposite directions.

Recall that: $\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

Subtracting we obtain:

$$\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

Therefore any single component $y_j(x,t)$ of the function $y_n(x,t)$

$$y_j(x,t) = \sin\frac{j\pi x}{L}\sin\frac{j\pi}{L}\sqrt{\frac{T}{\rho}}t = \frac{1}{2}\cos\frac{j\pi}{L}\left(x - \sqrt{\frac{T}{\rho}}t\right) - \frac{1}{2}\cos\frac{j\pi}{L}\left(x + \sqrt{\frac{T}{\rho}}t\right)$$

Wave traveling to the right. Wave traveling to the left.

Where the wave velocity V is $V = \sqrt{\frac{T}{\rho}}$.

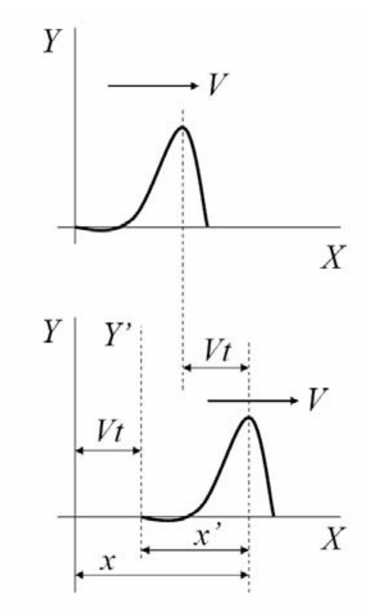


Figure 2

Figure 2 shows how the equation in time is set up for advancing pulse. The upper diagram shows the pulse at t = 0 given by the equation $y = f(x)$. The lower diagram shows the same pulse at time $t = t$ having advanced a distance Vt without changing shape. A new axis Y' is constructed, displaced a distance $x = Vt$ to the right, x' being the new coordinate of any point referred to the new origin. The equation $f(x)$ at time t in terms of x' is the same as the equation at $t = 0$ in terms of x, or $y = f(x')$ for $t = t$. However:

$$x' = x - Vt \quad therefore \quad y = f(x - Vt)$$

Initial conditions

Suppose the I.C.s are given by a function $y(x)$ such that for $t = 0$:

$$y(x,0)=\frac{2xd}{L} \qquad 0\le x\le \frac{L}{2}$$
$$y(x,0)=\frac{2d}{L}\left(L-x\right) \quad \frac{L}{2}\le x\le L$$
$$\frac{dy(x,0)}{dt}=0$$

Applying the velocity boundary conditions for t = 0 to (17) we obtain:

$$\frac{dy}{dt}=\sum_{n=1}^{\infty}B_n\frac{n\pi}{L}\sqrt{\frac{T}{\rho}}\cos\frac{n\pi x}{L}=0$$

Which satisfies the I.C.s by setting $B_n=0$. Solution of the problem requires determining the constants A_n so that that the initial conditions are satisfied:

$$\begin{aligned}\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}&=\frac{2d}{L}x \qquad 0\le x\le \frac{L}{2}\\ &=\frac{2d}{L}\left(L-x\right) \quad \frac{L}{2}\le x\le L\end{aligned} \tag{18}$$

multiplying both sides of (18) by $\sin\frac{m\pi x}{L}$ and integrating between 0 and L we obtain:

$$\int_0^L\left(\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}\right)\sin\frac{m\pi x}{L}dx=\int_0^{L/2}\frac{2d}{L}\sin\frac{m\pi x}{L}dx+\int_{L/2}^{L}\frac{2d}{L}\left(L-x\right)\sin\frac{m\pi x}{L}dx \tag{19}$$

Integrating (19) term by term:

$$A_m\frac{L}{2}=\frac{2d}{L}\left[\int_0^{L/2}x\sin\frac{m\pi x}{L}dx+\int_{L/2}^{L}(L-x)\sin\frac{m\pi x}{L}dx\right] \tag{20}$$

These integrals can be evaluated by considering that:

$$\int x\sin xdx=\sin x-x\cos x \quad and \quad \int x\sin axdx=\frac{1}{a^2}\int ax\sin axdax=\frac{1}{a^2}\int\theta\sin\theta d\theta \tag{21}$$

where $ax=\theta$. In (20), upon integration and evaluation at the limits the only non-zero terms exist at $L/2$, of which there are two, with same sign. Therefore, in (21) we can set $a=m\pi/L$, then

$$A_m = \frac{8d}{m^2\pi^2}\sin\frac{m\pi x}{L}$$

which vanishes when m is even and when it is odd the sine term oscillates between the values ± 1.

The product solution therefore is:

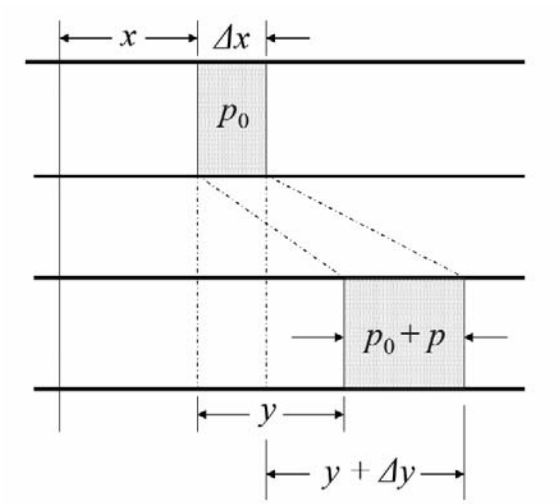
$$y(x,t) = \sum_{m=1}^{\infty}\frac{8d}{m^2\pi^2}\sin\frac{m\pi x}{L}\cos\frac{m\pi}{L}\sqrt{\frac{T}{\rho}}t \quad m = 1,3,5,... \tag{22}$$

BENG 221
Lecture 18
M. Intaglietta

Transmission of waves in gases. Sound.

Strings present the transmission of transverse waves. In gases waves are transmitted longitudinally. We will analyze the transmission of waves in tube where gas displacements are made by a piston. Moving the piston creates a compression that travels forward. If the piston is quickly retracted then there is a wave of rarefaction that also travels along the tube.

Consider an element of gas in the tube located between x and $x+\Delta x$ where the gas has an equilibrium pressure p_0. As the wave advances the element of gas oscillates about its equilibrium position. The coordinate y is used to describe displacements of gas from its equilibrium position. The displacement of the left side of the element of gas has coordinate y and that on the right side $y+\Delta y$. Pressure on the left side is p and on the right side is $p+\Delta p$. For a very thin slice pressure in the displaced gas is $p+p_0$, which is also the pressure on the left side face, and the pressure on the right side face is $p+p_0+\Delta p$. The forces acting on the element of gas are obtained by multiplying by the area of the tube A. The net restoring force acting on the displaced gas is $-\Delta pA$. If ρ_0 is the density of the gas at the equilibrium pressure p_0 then the mass of element is $\rho_0 A\Delta x$ leading to the equation of motion:



Note that x gives the position of the gas molecules at rest (therefore while p_0 is uniform in Δx) while y gives the position of displaced molecules and p is not uniform in Δy. In the case illustrated since $\Delta y > \Delta x$ we are dealing with a rarefaction wave.

$$\left((p+p_0)-(p+p_0+\Delta p)\right)A = -\Delta pA = \rho_0 A\Delta x\frac{d^2y}{dt^2}$$

$$\frac{d^2 y}{dt^2} = -\frac{1}{\rho_0}\frac{\Delta p}{\Delta x}$$

and at the limit for very small Δx:

$$\frac{\partial^2 y}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad (23)$$

The volume in its equilibrium position is $A\Delta x$. In the displaced position the coordinate of the right face is $x + \Delta x + y + \Delta y$ while the coordinate of the left face is $x + y$. Therefore the length of the displaced element is given by the difference of these two coordinates or $\Delta x + \Delta y$ and the change in length, and therefore the change in volume is $A\Delta y$.

Consider the general definition of compressibility k:

$$k = -\frac{1}{\textit{original volume}}\frac{\textit{change in volume}}{\textit{change in pressure}}$$

Note that the compressibility of a gas can be derived from the perfect gas equation $pV = RT$ where

$$\frac{dV}{dp} = -\frac{RT}{p^2} \quad \textit{and} \quad k = -\frac{1}{V}\frac{dV}{dp} = \frac{pRT}{RTp^2} = \frac{1}{p}$$

Referring this definition to our development:

$$k = -\frac{1}{A\Delta x}\frac{A\Delta y}{\left(\left(p_0 + p\right) - p_0\right)} = -\frac{\Delta y}{p\Delta x}$$

therefore:

$$p = -\frac{1}{k}\frac{\Delta y}{\Delta x}$$

and in the limit:

$$p = -\frac{1}{k}\frac{\partial y}{\partial x}$$
(24)

and in view of (23)

$$\frac{\partial p}{\partial x} = -\frac{1}{k}\frac{\partial^2 y}{\partial x^2}$$

Therefore substituting in (23) we are led to the one dimensional wave equation for the transmission of longitudinal perturbations:

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{k\rho_0}\frac{d^2 y}{dx^2}$$

The velocity of propagation, by analogy to the wave equation for strings (lateral displacements) is given by:

$$v = \sqrt{\frac{1}{k\rho_0}}$$

The bulk modulus *B* is the reciprocal of the compressibility, in other words, the pressure required to induce a volume change relative to the total volume. This quantity is the equivalent to the Young's modulus *Y* for linear changes (stress required o induce a change in strain). Therefore a general expression for the velocity at which waves travel in a materials is:

$$v = \sqrt{\frac{B}{\rho_0}}$$

Pressure variation in a sound wave

From the development of the propagation velocity of lateral displacement (waves) in a string we found that a disturbance is propagated with a velocity v, where in this case *L* = wave length, and *A* = displacement amplitude

$$y = A\cos\frac{2\pi n}{L}(x - vt)$$

If we know the displacement as a function of time $y(x,t)$ we can compute the pressure by differentiating with respect to *x* since:

$$p = -\frac{1}{k}\frac{\partial y}{\partial x}$$

in view of (24) which leads to:

$$\frac{dy}{dx} = -\frac{2\pi A}{L} \sin \frac{2\pi}{L}(x - \nu t)$$

and therefore:

$$p = \frac{2\pi A}{kL} \sin \frac{2\pi}{L}(x - \nu t)$$

Since $\nu = \sqrt{\frac{1}{k\rho_0}}$

$$p = \left[\frac{2\pi\rho_0 \nu^2 A}{L} \right] \sin \frac{2\pi}{L}(x - \nu t) \qquad (25)$$

The term within brackets represents the maximal pressure amplitude *P* while *A* is the maximal displacement.

Wave velocity & thermodynamics of perfect gases

The previous equation (25) shows that:

$$\frac{\partial p}{\partial \rho_0} = \frac{2\pi A}{L} \nu^2 \sin \frac{2\pi}{L}\left(x - \nu t\right)$$
(26)

therefore using the thermodynamic relation:

$$pV = nRT = n\frac{M}{M}RT \quad \text{where} \quad \frac{nM}{V} = \rho \quad \text{then} \quad p = \frac{\rho RT}{M}$$

therefore in terms of maximum values (26) is also equal to:

$$\frac{\partial p}{\partial \rho} = \frac{RT}{M} = \nu^2$$

a result derived by Newton, which underestimates the actual speed by about 15%. The more correct formulation was give by Laplace, who realized that the compression and relaxation in the sound wave is too rapid for allowing constant temperature (isothermal conditions), and that the actual conditions were adiabatic, i.e., no heat transfer due to the high speed at which compression and rarefaction occur, leading to the expression:

$$\nu = \sqrt{\gamma \frac{RT}{M}}$$

where $\gamma = \frac{C_p}{C_v}$

which is the ratio of specific heat at constant pressure vs. constant volume, usually about 1.4.

Pressure dispersion

The analytical derivation does not include a mechanism for the attenuation of the pressure amplitude, which occurs due to refraction and absorption of the pressure wave. This can be accounted for the expression:

$$A = A_o e^{-\alpha x}$$

where α is a parameter that characterizes the viscous effects in the medium, or the conversion of mechanical energy in the wave into thermal energy.

Pressure waves that originate from a point source decay naturally at the rate of -60 log *R* db, where *R* is the ratio of radial distances, and for cylindrical sources at the rate of -40 log *R*.

The dispersion of pressure can also be described by the diffusion equation:

$$\frac{\partial p}{\partial t} = K\nabla^2 p$$

Intensity of sound waves

Waves propagate energy. The intensity *I* of a traveling wave is defined as the average rate energy is transported by the wave per unit area across a surface perpendicular to the direction of propagation. Also intensity is the average power transported per unit area. The energy associated with a travelling wave is in part potential, associated with the compression of the medium, and kinetic relate to particle velocity. By analogy to the dynamics and energy distribution of the spring mass system, the total energy is constant in time (no dissipation) ad we can calculate intensity by considering only pressure effects.

Work done in the compression process is:

$$W = -\int pdv$$

and introducing the definition for compressibility:

$$dv = -kv_0 dp\partial \quad \text{therefore} \quad W = -kv_0 \int pdp$$

Integrating between 0 and the maximum pressure change *P* defines the pressure energy per unit volume, which is the same as the total energy per unit volume:

$$\frac{W}{v_0} = \frac{1}{2} kP^2$$

The sound energy in the volume traveling with a wave, or energy crossing per unit area per unity time equals the energy in the volume element $Av\Delta t$ divided by *A*:

$$I = \frac{1}{2} kP^2 v = \frac{P^2}{2\rho v}$$