

Lecture 2

Linear Time-Invariant Systems

References

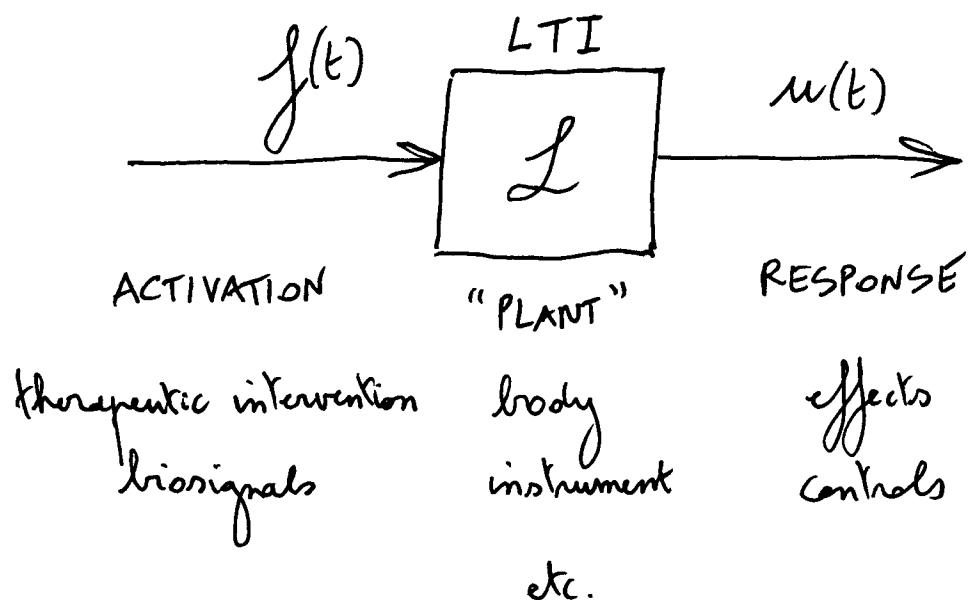
http://en.wikipedia.org/wiki/LTI_system_theory

<https://en.wikipedia.org/wiki/Convolution>

https://en.wikipedia.org/wiki/Bode_plot

LINEAR TIME-INVARIANT (LTI) SYSTEMS

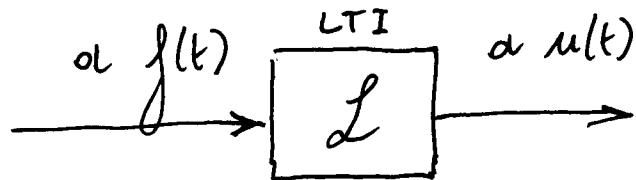
→ Modeling, analysis, and control of dynamics in bioengineering



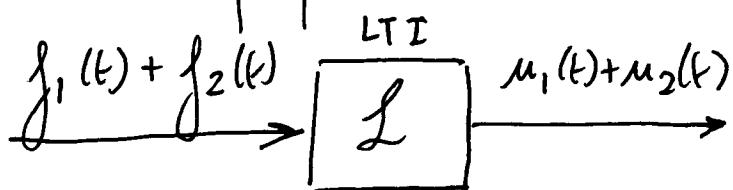
e.g.:	insulin	blood	glucose level
	force	limb	motion
	pacemaker signal	heart	ECG rhythm

LINEARITY:

invariance to scaling:

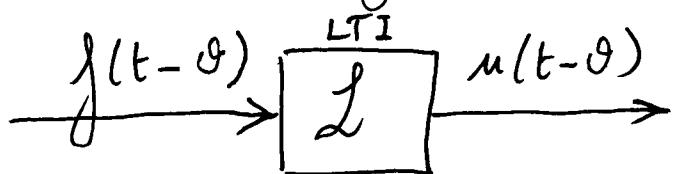


invariance to superposition:

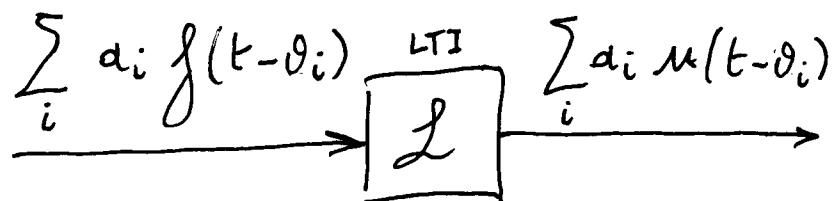


TIME INVARIANCE:

invariance to time shifts:



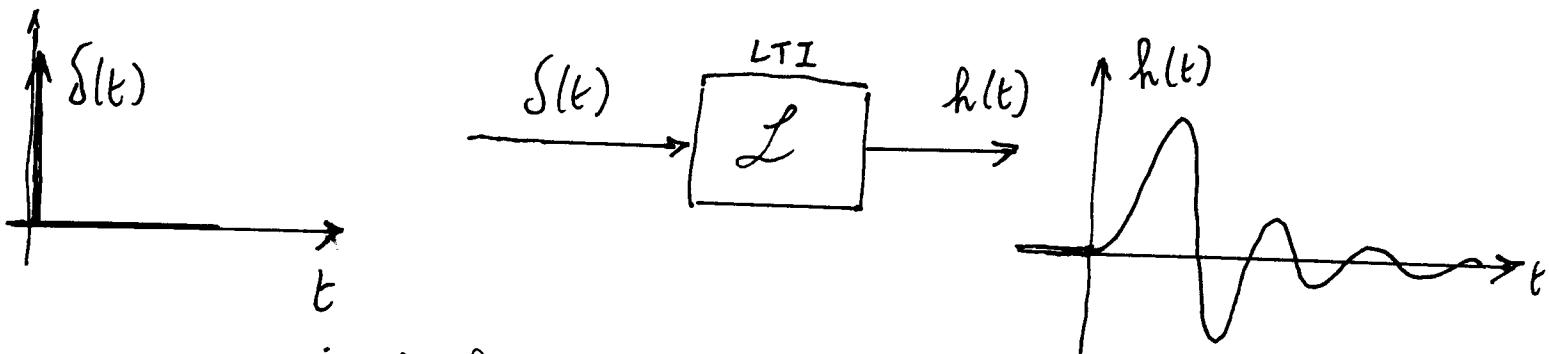
More generally:



\Rightarrow The dynamics of the system is completely characterized by its response to a particular activation, e.g.:

IMPULSE, SINGLE TONE, EXPONENTIAL

IMPULSE RESPONSE AND CONVOLUTION :



DELTA-DIRAC impulse function

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\int_{-\epsilon}^{+\epsilon} \delta(t) dt = 1$$

IMPULSE RESPONSE
of the system

↓ LTI

$$f(t) = \int_{-\infty}^{+\infty} f(\theta) \delta(t-\theta) d\theta$$

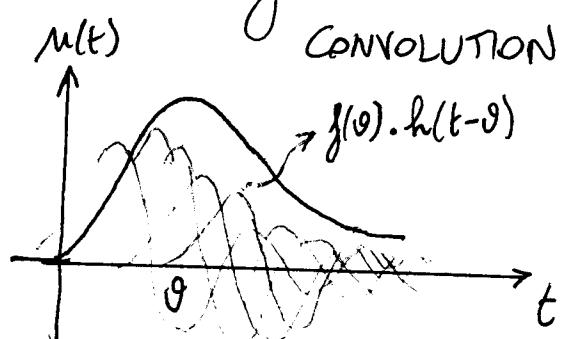
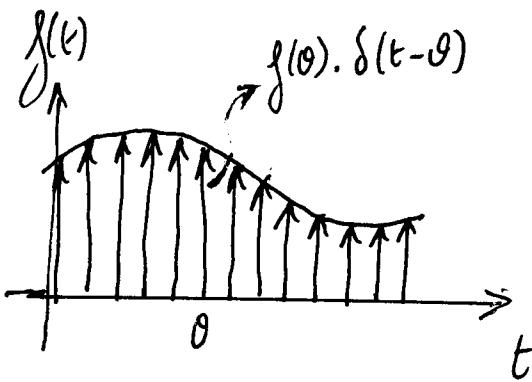


$$u(t) = \int_{-\infty}^{+\infty} f(\theta) h(t-\theta) d\theta$$

ACTIVATION $f(t)$

IMPULSE
RESPONSE
 $h(t)$

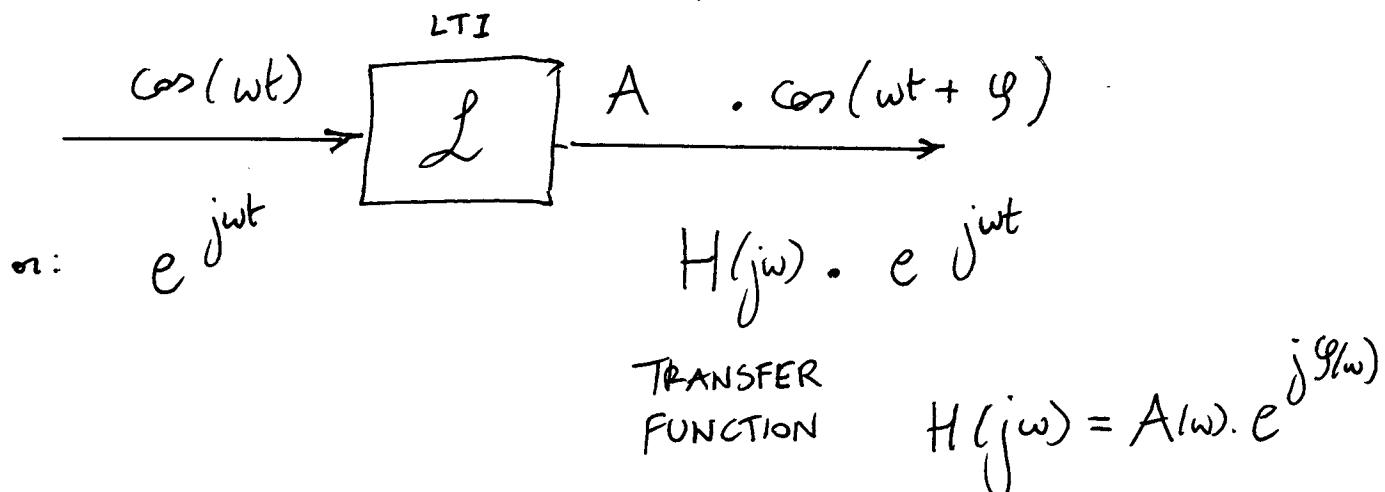
RESPONSE $u(t)$
= $f(t) \circledast h(t)$



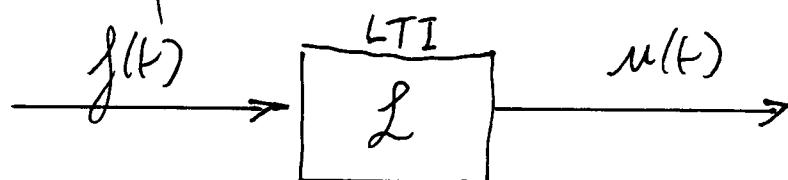
⇒ The system response to any input is the CONVOLUTION of that input with the IMPULSE RESPONSE.

HARMONIC RESPONSE AND FOURIER TRANSFER FUNCTION:

The LTI response to a sinusoidal input is a SCALED, PHASE-SHIFTED version of that input:



Fourier analysis of ANY signal $f(t)$ decomposed in its spectral components:



$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

FOURIER TRANSFORM

of input

$$U(j\omega) = \int_{-\infty}^{+\infty} u(t) e^{-j\omega t} dt$$

FOURIER TRANSFORM

of output

$$\Rightarrow U(j\omega) = H(j\omega) \cdot F(j\omega)$$

$$\text{where } H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$

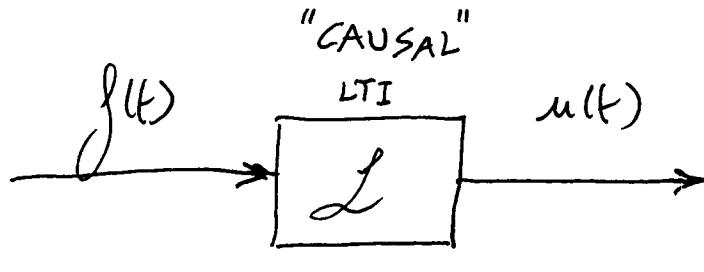
TRANSFER FUNCTION is the FOURIER TRANSFORM of the IMPULSE RESPONSE!

Indeed :

$$u(t) = h(t) \circledast f(t) \quad \text{CONVOLUTION}$$
$$= \int_{-\infty}^{+\infty} f(\theta) \cdot h(t-\theta) d\theta$$

$$\Rightarrow U(j\omega) = \int_{-\infty}^{+\infty} u(t) e^{-j\omega t} dt$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\theta) \cdot h(t-\theta) d\theta e^{-j\omega t} dt$$
$$= \underbrace{\int_{-\infty}^{+\infty} f(\theta) e^{-j\omega \theta} d\theta}_{F(j\omega)} \cdot \underbrace{\int_{-\infty}^{+\infty} h(t-\theta) e^{-j\omega(t-\theta)} dt}_{H(j\omega)}$$

LAPLACE TRANSFER FUNCTION AND TRANSIENT RESPONSE:



CAUSAL :

$$h(t) = 0 \text{ for } t < 0$$

$$F(s) = \int_0^{+\infty} f(t) e^{-st} dt$$

LAPLACE TRANSFORM

of input

$$U(s) = \int_0^{+\infty} u(t) e^{-st} dt$$

LAPLACE TRANSFORM

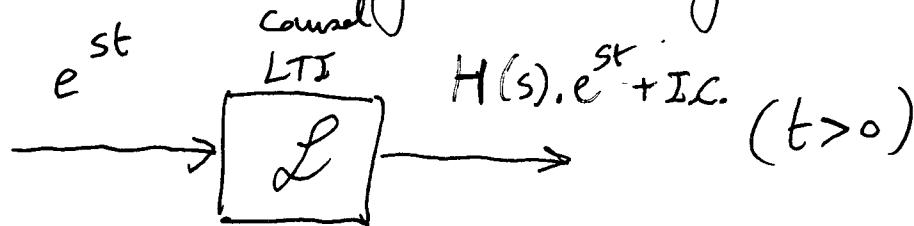
of output

$$\Rightarrow U(s) = H(s) \cdot F(s)$$

$$\text{where } H(s) = \int_0^{+\infty} h(t) e^{-st} dt$$

LAPLACE TRANSFER FUNCTION is the LAPLACE TRANSFORM
of the IMPULSE RESPONSE

\Rightarrow Equivalence $s = j\omega$ for causal systems
TRANSIENT RESPONSE for $t > 0$ from INITIAL CONDITIONS
(I.C.)



SOLUTION TO LTI ODES THROUGH LAPLACE TRANSFORMS:

- Use FOURIER transforms when the ODE problem has no initial conditions (or finite initial conditions at $t \rightarrow -\infty$)
- Use LAPLACE transforms when the ODE problem formulates initial conditions at $t=0$ (or at t_0 , through translation of the time coordinate $t' = t - t_0$)

$$\mathcal{L} \left(u(t), \frac{du}{dt}, \frac{d^2u}{dt^2}, \dots, \frac{d^n u}{dt^n} \right) = f(t) \quad \text{with} \quad \begin{cases} u(0) = u_0 \\ \frac{du}{dt}(0) = u_0^{(1)} \\ \vdots \\ \frac{d^{n-1}u}{dt^{n-1}}(0) = u_0^{(n-1)} \end{cases}$$

↓
LINEAR
OPERATOR

$$f(t) \rightarrow F(s)$$

$$u(t) \rightarrow U(s)$$

$$\frac{du}{dt} \rightarrow sU(s) - u_0$$

$$\frac{d^2u}{dt^2} \rightarrow s^2U(s) - su_0 - u_0^{(1)}$$

$$\vdots$$

$$\frac{d^n u}{dt^n} \rightarrow s^n U(s) - s^{n-1}u_0 - s^{n-2}u_0^{(1)} - \dots - u_0^{(n-1)}$$

\Rightarrow Solving the ODE amounts to solving an algebraic equation in the s Laplace variable.

EXAMPLE: Damped spring-mass system under gravity

$$m \frac{d^2 u}{dt^2} + \gamma \frac{du}{dt} + k u = -mg$$

with $\begin{cases} u(0) = u_0 & \text{initial position} \\ \frac{du}{dt}(0) = v_0 & \text{initial velocity} \end{cases}$

$$\Rightarrow (m s^2 + \gamma s + k) U(s) - m s u_0 - m v_0 - \gamma u_0 = -mg \cdot \frac{1}{s}$$

LAPLACE transform of
a constant step @ $t=0$

$$\Rightarrow U(s) = \frac{u_0 s + \frac{\gamma}{m} u_0 + v_0 - g \cdot \frac{1}{s}}{s^2 + \frac{\gamma}{m} s + \frac{k}{m}} \rightarrow \text{"POLES"}$$

- Two poles :
- complex, conjugate → UNDERDAMPED
 - real, identical → CRITICALLY DAMPED
 - real, different → OVERDAMPED

Interpretation of "poles":

Partial fraction decomposition of $U(s)$:

$$U(s) = \frac{c_1}{s - \lambda_1} + \frac{c_2}{s - \lambda_2} + \dots \quad (\text{typically})$$

where $\lambda_1, \lambda_2, \dots$ are the POLES (roots of the denominator of $U(s)$).

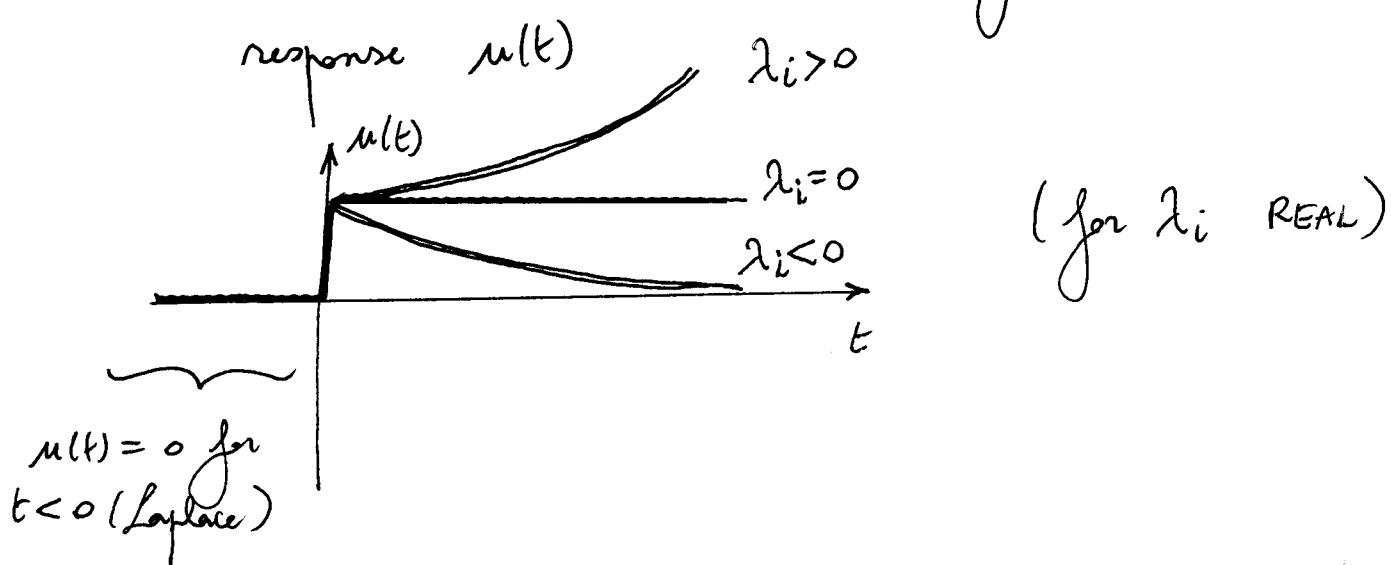
Inverse Laplace transform (see Laplace tables):

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots$$

\Rightarrow The POLES:

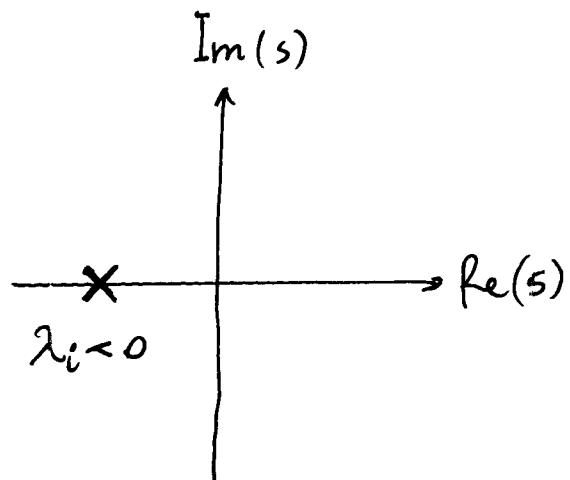
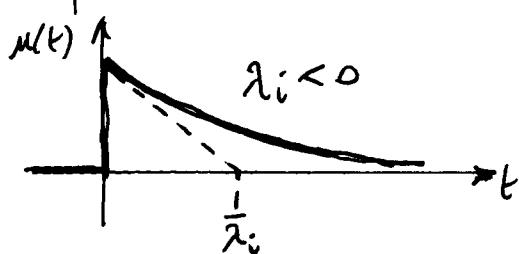
- are singular values in the Laplace transform, $s = \lambda_i$, where $U(s) \rightarrow \infty$

- determine the EIGENMODES $e^{\lambda_i t}$ of the transient



A LTI system is STABLE when all poles have a negative real component: $\operatorname{Re}(\lambda_i) \leq 0$
 (strictly stable: < 0)

- REAL poles:



- COMPLEX poles come in complex conjugate pairs:

$$\lambda_i = \sigma \pm j\omega$$

$$\Rightarrow e^{\lambda_i t} = e^{\sigma t} \cdot \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$$

↓ ↓
 DAMPING OSCILLATION

