BEGN 221 Mathematical Methods in Bioengineering

Fall 2012

Midterm

NAME: ____

- Open book, open notes.
- 3 hour limit, in one sitting.
- Return hardcopy before closing by 11am, Thursday November 1, 2012.
- No communication on the midterm anytime before closing other than with instructor and TAs.
- No computers or internet during the midterm, except for access to posted class materials and contact with instructor and TAs.

Problem 1 (30 points): The rise and fall of a single bacterial population $x(t)$ and a single nutrient $y(t)$ in a petri dish over time t are modeled by the following set of ordinary differential equations:

$$\begin{aligned} \frac{d}{dt}x(t) &= g\,x(t) + c\,y(t) \\ \frac{d}{dt}y(t) &= -d\,y(t) - e\,x(t) + q(t) \end{aligned}$$

where g is the intrinsic bacterial growth rate, c is the nutrition induced bacterial growth rate, d is the intrinsic nutrient decay rate, e is the nutrient consumption rate, and $q(t)$ is the spontaneous nutrient source generation over time. At time $t = 0$ the nutrient is fully depleted and the bacterial population is at an initial level x_0.

1. (5 points): Write the initial conditions. Are these sufficient to solve for a unique solution?

2. (10 points): Find the Laplace transform $\tilde{x}(s)$ of the solution for the bacterial population $x(t)$ from the initial conditions. Under what condition on the parameters $g, c, d,$ and e is the bacteria-nutrient system stable?

3. (5 points): Find the Fourier transfer function $H(j\omega)$ of the system with source input $q(t)$ and bacterial output $x(t)$.

4. (10 points): Find the solution $x(t)$ from initial conditions and zero source $q(t) = 0$ for the following values of the constants: $g = 0$, $c = 1\ s^{-1}$, $d = 2\ s^{-1}$, and $e = 1\ s^{-1}$. Make sure to indicate the units.

Problem 2 (30 points): Consider the following homogeneous partial differential equation with homogeneous boundary conditions:

$$\frac{\partial}{\partial t}u(x,t) = D\,\frac{\partial^2}{\partial x^2}u(x,t) \qquad \text{with} \qquad \left\{ \begin{array}{rcl} u(x,0) & = & g(x) \\ u(0,t) & = & 0 \\ \frac{\partial}{\partial x}u(L,t) & = & 0 \end{array} \right. \tag{1}$$

This partial differential equation is approximated using finite differences as:

$$u(x,t+\Delta t) = u(x,t) + \eta\,\left(u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)\right) \tag{2}$$

or, equivalently, evaluated on the grid as sequences $u_i[n] = u(i\Delta x, n\Delta t)$ for integer values of i and n:

$$u_i[n+1] = u_i[n] + \eta\,\left(u_{i+1}[n] - 2u_i[n] + u_{i-1}[n]\right), \qquad i = 1,\ldots N-1; n = 0,\ldots\infty. \tag{3}$$

1. (10 points): For length $L = 1\ m$, diffusivity $D = 0.01\ m^2/s$ and update constant $\eta = 0.1$, find the grid constants Δx and Δt such that your finite difference approximation resolves at least 100 points ($i = 0,\ldots N = 100$) over the $[0, L]$ interval.

2. (5 points): Write the finite difference approximation to the initial conditions at $t = 0$, in terms of $u_i[0]$, for given $g_i = g(i\Delta x)$.

3. (5 points): Write the finite difference approximation to the boundary condition at $x = 0$, in terms of $u_0[n]$.

4. (10 points): Write the finite difference approximation to the boundary condition at $x = L$, in terms of $u_N[n]$ and its neighbors on the grid.

5. **BONUS** (10 extra points, no partial credit–*only pursue this if you have time left after completing everything else*): Write the finite difference approximations to the non-homogeneous partial differential equation with non-homogeneous boundary conditions:

$$\frac{\partial}{\partial t}u(x,t) = D\,\frac{\partial^2}{\partial x^2}u(x,t) + q(x,t) \qquad \text{with} \qquad \left\{ \begin{array}{rcl} u(x,0) & = & g(x) \\ u(0,t) & = & h(t) \\ \frac{\partial}{\partial x}u(L,t) & = & f(t) \end{array} \right. \tag{4}$$

in terms of $u_i[n]$ for given g_i as defined above, and given $q_i[n] = q(i\Delta x, n\Delta t)$, $h_0[n] = h(n\Delta t)$, and $f_0[n] = f(n\Delta t)$. Indicate the range of valid indices i and n for each equation including initial/boundary conditions.

Problem 3 (40 points): Oxygen diffuses in a slice preparation of brain tissue of thickness $L = 1\ mm$ with diffusivity $D = 1mm^2 \,/\, s$. The slice is perfused with oxygenated solution generating constant and equal (opposing) influx of oxygen $\Phi_{ox} = 0.1\ \mu mol \,/\, mm^2 s$ on both sides, into the tissue. Oxygen is consumed by the tissue at a constant rate $R = 0.2\ \mu mol \,/\, mm^3 s$. At initial time $t = t_0$, the oxygen concentration is $u_0 = 1\ \mu mol \,/\, mm^3$ uniform inside the tissue.

1. (10 points): Write down the partial differential equation with initial and boundary conditions for oxygen concentration $u(x,t)$ in the tissue. Verify consistency in the units.

2. (10 points): Solve a modified version of this problem, for the homogeneous partial differential equation with homogeneous boundary conditions and initial conditions $u(x, t_0) = \delta(x - x_0)$. What does this solution represent, and why is finding this solution useful in solving the original problem?

3. (10 points): Find a particular (steady-state) solution $u_p(x)$ to the original problem (but disregarding the initial conditions) that does not depend on time. Find the conditions on the parameters $L, D, \Phi_{ox},$ and R for such solution to exist. Is the solution unique? Why is finding this solution useful in solving the original problem?

4. (10 points): Now find the full solution $u(x,t)$ to the original problem from the initial conditions.