BENG 221 Mathematical Methods in Bioengineering

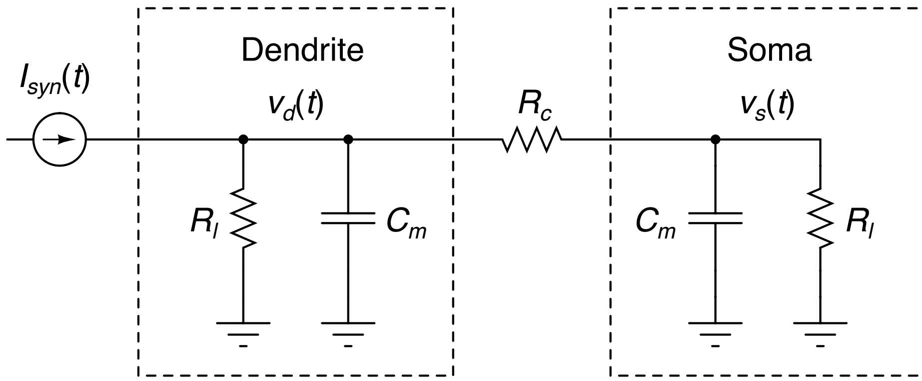
Fall 2013

Midterm

NAME: ______________________________

- Open book, open notes.
- 3 hour limit, in one sitting.
- Return hardcopy before closing by 11am, Thursday October 31, 2013.
- No communication on the midterm anytime before closing other than with instructor and TAs.
- No computers or internet during the midterm, except for access to posted class materials and contact with instructor and TAs.

Problem 1 (30 points): Consider a two-compartment model of a neuron with dendrite and soma, shown below. The dendrite and soma both have membrane capacitance C_m and leak resistance R_l to grounded extracellular space (at zero potential), and are coupled by a cable resistance R_c. Both dendrite and soma are initially at zero rest potential, $v_d(0) = v_s(0) = 0$. A pulse of synaptic current $I_{syn}(t) = Q_0\,\delta(t)$ with charge Q_0 enters the dendrite at time zero.



1. (5 points): Write the differential equations and initial conditions governing the dynamics of the dendrite potential $v_d(t)$ and soma potential $v_s(t)$.

2. (15 points): Find the soma potential $v_s(t)$ over time from the initial conditions. You may use Laplace or any method of your choice. HINT: In the derivations it is easier to express the problem in terms of conductances rather than resistances, *i.e.,* leak conductance $g_l = 1/R_l$ and cable conductance $g_c = 1/R_c$.

3. (10 points): Find the minimum synaptic charge Q_0 to be injected in the dendrite in order for the soma potential to reach a given spike threshold V_{th}. At what time does it cause the soma to spike?

Problem 2 (40 points): Consider the following homogeneous partial differential equation (PDE) with homogeneous boundary conditions and homogeneous initial velocity condition:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t) + \frac{1}{D}\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) \quad \text{with} \quad \left\{ \begin{array}{rcl} u(x,0) & = & g(x) \\ \frac{\partial}{\partial t}u(x,0) & = & 0 \\ u(0,t) & = & 0 \\ \frac{\partial}{\partial x}u(L,t) & = & 0 \end{array} \right. \tag{1}$$

1. (5 points): What physical quantities do D and c represent, and what are their units? Interpret two limiting cases of this PDE: one for which $D \to \infty$, and the other for which $c \to \infty$.

2. (25 points): Find the homogeneous solution to the PDE by separation of variables. As usual, express your solution in the form $u(x,t) = \sum_i c_i\, \phi_i(x)\, G_i(t)$, where $\phi_i(x)$ and $G_i(t)$ are solutions to resulting ODEs with eigenvalues λ_i subject to resulting boundary conditions, and where the coefficients c_i are determined from the initial conditions $g(x)$. HINT: The twist in this problem is that *all* eigenmodes $\phi_i(x)G_i(t)$ *must* satisfy the additional homogeneous initial *velocity* condition $\frac{\partial}{\partial t}u(x,0) = 0$, *i.e.,* $\frac{d}{dt}G_i(0) = 0$. You may assume that all $G_i(t)$ have complex conjugate poles in the Laplace domain.

3. (5 points): Find the range of the length L, for given parameters D and c, that guarantees that the poles for each of the $G_i(t)$ are complex conjugate, as assumed above.

4. (5 points): Based on your homogenous solution, write the Green's function $G(x, t; x_0, t_0)$.

Problem 3 (30 points): Consider heat conduction in tissue of thickness L between skin (at $x = 0$) and bone (at $x = L$). The heat conductivity of the tissue is K_0, its specific heat is c, and its mass density is ρ, all uniform across the tissue. Skin conduction to the environment maintains the skin at constant temperature T_e. In contrast, the bone is thermally insulating. Heat generation in the tissue due to nonuniform metabolic energy consumption depends on distance from the skin x as $Q(x,t) = Q_0 \sin(\pi x/2L)$ for all $t < 0$. At time $t = 0$, the oxygen supply in the bloodstream to the tissue is terminated and all heat generation stops.

1. (10 points): Find the steady state temperature distribution $u_0(x)$ in the tissue before the termination of the oxygen supply.

2. (5 points): Write down the partial differential equation with initial and boundary conditions for temperature $u(x,t)$ in the tissue after termination of the oxygen supply ($t \geq 0$, with initial conditions expressed at $t = 0$).

3. (15 points): Solve for the dynamics of $u(x,t)$ for $t \geq 0$.