BENG 221 Mathematical Methods in Bioengineering

Fall 2014

Midterm

NAME: SOLUTIONS

- Open book, open notes.
- 80 minutes limit (end of class).
- No communication other than with instructor and TAs.
- No computers or internet, except for access to posted class materials.

Problem 1 (30 points): Short answer problems. Provide brief explanations (no lengthy derivations!) for each problem.

1. (5 points): A random walk process with random step size of standard deviation Δx and step time interval Δt is characterized with a diffusivity D. How does the diffusivity D change as the random step size increases by a factor two (*i.e.*, Δx becomes $2\Delta x$)?

D increases 4x (D becomes 4D) because diffusivity scales as $\frac{\Delta x^2}{\Delta t}$

2. (5 points): Conversely, how does the diffusivity D change as the step time interval increases by a factor two (*i.e.*, Δt becomes $2\Delta t$)?

D halves (D becomes $\frac{D}{2}$) similarly

3. (10 points): Find the radial frequency ω of wave oscillations of a free electron with energy E.

$$\omega = \frac{E}{\hbar} = 2\pi \frac{E}{h}$$

from the time-dependent part $e^{-i\frac{E}{\hbar}t}$ in separation of variables of the Schrödinger equation $i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi = E\Psi$.

Also consistent with $E = \hbar\omega = h\nu$ (Planck)

4. (10 points): Find the maximum time step Δt beyond which Euler numerical integration of the ODE $dx/dt = -x/\tau$ becomes unstable (*i.e.*, gives unbounded results).

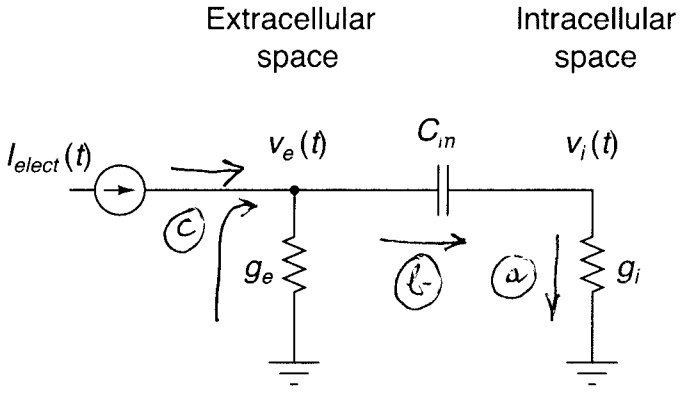
$$\Delta t_{max} = 2\tau$$

because $x(t+\Delta t) = x(t) + \Delta t\left(-\frac{x(t)}{\tau}\right) = \left(1 - \frac{\Delta t}{\tau}\right) x(t)$

thus $\left|\frac{x(t+\Delta t)}{x(t)}\right| = \left|1 - \frac{\Delta t}{\tau}\right| > 1$ for $\Delta t > 2\tau$

leading to unbounded $x(t)$ as $t \to \infty$

Problem 2 (25 points): Consider an electrically excitable cell as shown below. The cell has membrane capacitance C_m and leak conductance g_i. The extracellular space has leak conductance g_e. Both intracellular and extracellular potentials are initially zero. At time zero, a constant current electrode current $I_{elect}(t) = I_0$ is injected into the extracellular space.



current a = b = c

1. (5 points): Write the differential equations and initial conditions governing the dynamics of the intracellular and extracellular potentials, $v_i(t)$ and $v_e(t)$.

$$\underset{a}{g_i v_i} = \underset{b}{C_m \frac{d}{dt}(v_e - v_i)} = -g_e v_e \underset{c}{+} I_{elect}(t)$$

where $I_{elect}(t) = I_0$ $(t \geq 0)$

I.C. : $v_i(0) = v_e(0) = 0$

2. (15 points): Find the intracellular potential $v_i(t)$ over time. You may use Laplace or any method of your choice.

Laplace :

$v_i(t) \rightarrow \tilde{v}_i(s)$ $\quad \frac{d}{dt} v_i \rightarrow s\tilde{v}_i$ $\quad (v_i(0)=0)$

$v_e(t) \rightarrow \tilde{v}_e(s)$ $\quad \frac{d}{dt} v_e \rightarrow s\tilde{v}_e$ $\quad (v_e(0)=0)$

$I_{elect}(t) = I_0 \rightarrow I_0 \cdot \frac{1}{s}$
$(t \geq 0)$

$$g_i \tilde{v}_i = C_m s (\tilde{v}_e - \tilde{v}_i) = -g_e \tilde{v}_e + \frac{1}{s} I_0$$

Eliminating $\tilde{v}_e$: $g_e \tilde{v}_e = -g_i \tilde{v}_i + \frac{1}{s} I_0$

$$\Rightarrow g_i g_e \tilde{v}_i = C_m I_0 - (g_i + g_e) C_m s \tilde{v}_i$$

$$\tilde{v}_i = \frac{C_m I_0}{(g_i+g_e) C_m s + g_i g_e} = \frac{I_0}{g_i+g_e} \cdot \frac{1}{s + \frac{g_i g_e}{g_i+g_e} \cdot \frac{1}{C_m}}$$

Inverse Laplace : $\frac{1}{s+a} \rightarrow e^{-at}, \; t \geq 0$

$$\Rightarrow v_i(t) = \frac{I_0}{g_i+g_e} \cdot e^{-\frac{g_i g_e}{g_i+g_e} \cdot \frac{1}{C_m} \cdot t}, \; t \geq 0$$

3. (5 points): Find the minimum level of electrode current I_0 to be injected in order for the intracellular potential $v_i(t)$ to reach the threshold V_{th} at which the cell generates an action potential. At what time does the action potential happen? Does the duration of the electrode current matter, and how?

$v_i(t)$ is maximum at $t \overset{(*)}{=} 0$, regardless of duration of electrode current

$\Rightarrow$ Action potential happens at $t \overset{(*)}{=} 0$ if

$$v_i(0) \overset{(*)}{=} \frac{I_0}{g_i+g_e} \geq V_{th}, \text{ or } I_0 \geq (g_i+g_e) V_{th}$$

$$\left(I_{0\,min} = (g_i+g_e) V_{th} \right)$$

(*) AFTER the step in current I_0 at time $t=0$.

Problem 3 (45 points): An athlete initially at rest starts to exercise. The body is covered with thermally insulating material. Underneath the skin ($x = 0$) is muscle tissue of thickness L, interfacing on the other end ($x = L$) with vasculature. The thermal conductivity of the muscle tissue is K_0, and the vasculature conducts heat to maintain the tissue interface at a constant temperature T_0. Specific heat of the muscle tissue is c, and mass density is ρ. Once starting to exercise ($t \geq 0$), the athlete burns calories (Joules) uniformly in the muscle tissue at constant rate, with heat generation $Q(x,t) = Q_0$.

1. (5 points): Write the partial differential equation governing temperature $u(x,t)$ in the muscle tissue. Express initial and boundary conditions.

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q_0$$

$$\text{I.C.:} \quad u(x,0) = T_0$$

$$\text{B.C.:} \quad \begin{cases} \frac{\partial u}{\partial x}(0,t) = 0 \\ u(L,t) = T_0 \end{cases}$$

2. (10 points): Solve for a particular solution $u_p(x)$ for the temperature in the tissue at steady state.

Steady state: $\frac{\partial u}{\partial t} \to 0$ for $t \to \infty$

$$\Rightarrow \quad K_0 \frac{d^2 u_p}{dx^2} + Q_0 = 0 \quad \text{with B.C.} \begin{cases} \frac{du_p}{dx}(0) = 0 \\ u_p(L) = T_0 \end{cases}$$

$$\frac{du_p}{dx} = -\frac{Q_0}{K_0} \int dx = -\frac{Q_0}{K_0} x + \cancel{a}$$

$$\text{B.C.:} \quad \frac{du_p}{dx}(0) = a = 0$$

$$u_p = -\frac{Q_0}{K_0} \int x\,dx = -\frac{Q_0}{2K_0} x^2 + b$$

$$\text{B.C.:} \quad u_p(L) = -\frac{Q_0}{2K_0} L^2 + b = T_0$$

$$\Rightarrow \quad u_p(x) = T_0 - \frac{Q_0}{2K_0}(x^2 - L^2)$$

3. (5 points): Write the homogeneous problem, with homogeneous partial differential equation and boundary conditions. You may combine constants into a thermal diffusivity D.

$$\frac{\partial u_H}{\partial t} = D \frac{\partial^2 u_H}{\partial x^2} \quad \text{with} \quad D = \frac{K_0}{c\rho}$$

$$\text{B.C.:} \begin{cases} \frac{\partial u_H}{\partial x}(0,t) = 0 & \text{(FLUX)} \\ u_H(L,t) = 0 & \text{(VALUE)} \end{cases}$$

4. (10 points): Find the eigenmode decomposition for the general solution of the homogenous problem.

Separation of variables : $u_H(x,t) = \phi(x) \cdot G(t)$

- TIME : $\frac{dG}{dt} = -D\lambda G \Rightarrow G(t) = \underbrace{G(0)}_{=1} \cdot e^{-D\lambda t}$

- SPACE : $\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \Rightarrow \phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

B.C. @ $x=0$: $\frac{d\phi}{dx}(0) = 0 \Rightarrow B = 0$

B.C @ $x=L$: $\phi(L) = 0 \Rightarrow \cos(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda} = (n+\frac{1}{2})\frac{\pi}{L}$

$n = 0, 1, 2, \ldots$

superposition

$$\Rightarrow u_H(x,t) = \sum_{n=0}^{\infty} A_n \cos\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right) e^{-D\left((n+\frac{1}{2})\frac{\pi}{L}\right)^2 t}$$

5. (15 points): Solve for the temperature in the tissue over time $u(x,t)$ from initial conditions.

$$u(x,t) = u_p(x) + u_H(x,t)$$

$$\text{I.C.: } u(x,0) = T_0 = u_p(x) + u_H(x,0)$$

$$= u_p(x) + \sum_{n=0}^{\infty} A_n \cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L \left(T_0 - u_p(x)\right)\cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)dx$$

$$= \frac{Q_0}{K_0}\int_0^L \left(\frac{x^2}{L} - L\right)\cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)dx$$

$$= -2L^2\frac{Q_0}{K_0}\frac{(-1)^n}{\left(\left(n+\frac{1}{2}\right)\pi\right)^3}$$

from lecture notes (or see next page)

$$\Rightarrow u(x,t) = T_0 - \frac{1}{2}\frac{Q_0}{K_0}(x^2 - L^2) - 2L^2\frac{Q_0}{K_0}\sum_{n=0}^{\infty}(-1)^n\frac{\cos\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{L}\right)}{\left(\left(n+\frac{1}{2}\right)\pi\right)^3}e^{-D\left(\left(n+\frac{1}{2}\right)\frac{\pi}{L}\right)^2 t}$$

$$A_n = \frac{Q_0}{K_0} \int_0^L \left(\frac{x^2}{L} - L\right) \cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right) dx$$

Partial integration twice:

$$A_n = \frac{Q_0}{K_0} \frac{L}{(n+\frac{1}{2})\pi} \int_0^L \left(\frac{x^2}{L} - L\right) d\left(\sin\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)\right)$$

$$= \frac{Q_0}{K_0} \frac{L}{(n+\frac{1}{2})\pi} \left(\underbrace{\left[\left(\frac{x^2}{L} - L\right) \sin\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)\right]_0^L}_{=0} - \int_0^L \frac{2x}{L} \sin\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right) dx \right)$$

$$= \frac{Q_0}{K_0} \left(\frac{L}{(n+\frac{1}{2})\pi}\right)^2 \int_0^L \frac{2x}{L} \, d\left(\cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)\right)$$

$$= \frac{Q_0}{K_0} \left(\frac{L}{(n+\frac{1}{2})\pi}\right)^2 \left(\underbrace{\left[\frac{2x}{L} \cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)\right]_0^L}_{=0} - \int_0^L \frac{2}{L} \cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right) dx \right)$$

$$= -\frac{Q_0}{K_0} \left(\frac{L}{(n+\frac{1}{2})\pi}\right)^3 \frac{2}{L} \underbrace{\left[\sin\left(\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}\right)\right]_0^L}_{(-1)^n} = -2L^2 \frac{Q_0}{K_0} \frac{(-1)^n}{\left((n+\frac{1}{2})\pi\right)^3}$$