BEENG 221 Mathematical Methods in Bioengineering

Fall 2016

Midterm

NAME: ____

- Open book, open notes.
- 80 minutes limit (end of class).
- No communication other than with instructor and TAs.
- No computers or internet, except for access to posted class materials.

Short Table of Laplace Transforms

$\mathbf{f(t)}$	$\hat{\mathbf{f}}(\mathbf{s}) = \mathcal{L}[\mathbf{f}](\mathbf{s})$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$

$\mathbf{f(t)}$	$\hat{\mathbf{f}}(\mathbf{s}) = \mathcal{L}[\mathbf{f}](\mathbf{s})$
$af(t)+bg(t)$	$a\hat{f}(s)+b\hat{g}(s)$
$f'(t)$	$s\hat{f}(s)-f(0+)$
$f^{(n)}(t)$	$s^n\hat{f}(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}\hat{f}(s)$
$tf(t)$	$-\hat{f}'(s)$
$t^n f(t)$	$(-1)^n\hat{f}^{(n)}(s)$
$\frac{1}{t}f(t)$	$\int_s^\infty \hat{f}^{(n)}(\sigma)\,d\sigma$
$e^{at}f(t)$	$\hat{f}(s-a)$
$f(t-a)H(t-a)$	$e^{-as}\hat{f}(s)$
$(f*g)(t)$	$\hat{f}(s)\hat{g}(s)$

Note: $\sinh(x) = \frac{e^x-e^{-x}}{2}$, $\cosh(x) = \frac{e^x+e^{-x}}{2}$

Problem 1 (30 points): Short answer problems. Provide brief explanations (no lengthy derivations!) for each problem.

1. (5 points): Is 1-D diffusion over a bounded interval $[0, L]$ linear space-invariant (LSI)?

2. (5 points): Is 1-D diffusion over a bounded interval $[0, L]$ linear time-invariant (LTI)?

3. (10 points): A numerical PDE solver is used to approximate a 1-D diffusion problem in $u(x, t)$ through the recursion:

$$u(x, t + \Delta t) = u(x, t) + \mu \cdot (u(x - \Delta x, t) - 2\, u(x, t) + u(x + \Delta x, t))$$

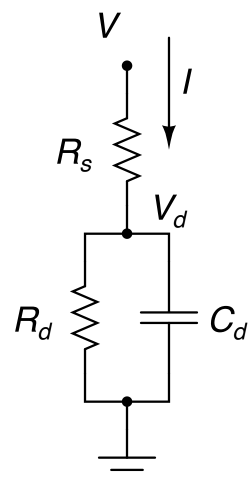
   where $\Delta x = 1$ mm, $\Delta t = 100$ ms, and $\mu = 0.1$. Find the effective diffusivity D.

4. (10 points): Consider the wave equation with following boundary and initial conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with:} \left\{ \begin{array}{rcl} u(0, t) & = & f(t); \\ \frac{\partial u}{\partial x}(L, t) & = & g(t); \\ u(x, 0) & = & u_0(x). \end{array} \right.$$

   What is missing in this problem statement to determine a unique solution in $u(x, t)$?

Problem 2 (25 points): An electrode in contact with electrolyte in tissue is modeled by the electrical circuit given below, with series resistance R_s, double-layer resistance R_d, and double-layer capacitance C_d. Initially the electrode is fully discharged, with zero voltage across the double-layer capacitance. At time zero a step in voltage $V(t) = V_0$ is applied across the electrode in order to stimulate the tissue.



1. (5 points): Write the differential equation and initial condition governing the dynamics of the double-layer voltage $V_d(t)$ as a function of the voltage $V(t)$ applied across the electrode.

2. (15 points): Find the current $I(t)$ flowing through the electrode over time. You may use Laplace or any method of your choice.

3. (5 points): Find the maximum stimulation current, and the time at which it is reached.

Problem 3 (45 points): An active thermal blanket of thickness L is placed over a patient suffering from hypothermia, in order to quickly and uniformly deliver heat to the body. The blanket is thermally insulating on both sides but actively pumps heat from the environment on one side, to the patient's body on the other side. In particular, assume the heat flux entering the blanket from the environment at one boundary ($x = 0$) is a constant Q_0, and the heat flux exiting the blanket to the patient at the other boundary ($x = L$) is the exact opposite $-Q_0$. The internal thermal conductivity of the blanket is K_0, and its specific heat and mass density are c and ρ. Initially the blanket is uniformly at temperature T_0.

1. (5 points): Write the partial differential equation governing temperature $u(x, t)$ in the blanket. Express initial and boundary conditions.

2. (10 points): Solve for a particular solution $u_p(x)$ for the temperature in the blanket at steady state.

3. (5 points): Write the homogeneous problem, with homogeneous partial differential equation and boundary conditions. You may combine constants into a thermal diffusivity D.

4. (10 points): Find the eigenmode decomposition for the general solution of the homogenous problem.

5. (15 points): Solve for the temperature in the blanket over time $u(x, t)$ from initial conditions.

6. **BONUS** (5 points): How much time does it take for the blanket with surface area A to raise the body temperature of the patient, with specific heat c_P and mass m_P, by ΔT?