BEGN 221 Mathematical Methods in Bioengineering

Fall 2017

Midterm

NAME: SOLUTIONS

- Open book, open notes.
- 80 minutes limit (end of class).
- No communication other than with instructor and TAs.
- No computers or internet, except for access to posted class materials.

Table 1: Laplace and Fourier Transforms

$u(t)$	$U(s)$	$u(t)$	$U(j\omega)$
$\delta(t)$	1	$\delta(t)$	1
1	$\frac{1}{s}$	1	$\frac{1}{j\omega}$
e^{-at}	$\frac{1}{s+a}$	e^{-at} for $t \geq 0$; 0 otherwise	$\frac{1}{j\omega + a}$
$u(t-t_0)$; $t_0 \geq 0$	$e^{-st_0}\, U(s)$	$u(t-t_0)$	$e^{-j\omega t_0}\, U(j\omega)$
$\frac{du}{dt}$	$s\, U(s) - u(0)$	$\frac{du}{dt}$	$j\omega\, U(j\omega)$
$\int_0^t u(t_0)\, dt_0$	$\frac{1}{s}\, U(s)$	$\int_{-\infty}^t u(t_0)\, dt_0$	$\frac{1}{j\omega}\, U(j\omega)$
$\int_0^t f(t_0)\, h(t-t_0)\, dt_0$	$H(s) \cdot F(s)$	$\int_{-\infty}^{+\infty} f(t_0)\, h(t-t_0)\, dt_0$	$H(j\omega) \cdot F(j\omega)$

Table 2: Green's Functions for Diffusion in 1-D

B.C. $x = 0$	$x = L$	$G(x, t; x_0, t_0)$ $t > t_0$
—	—	$\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) = \frac{1}{\sqrt{4\pi D(t-t_0)}}\ \exp(-\frac{(x-x_0)^2}{4D(t-t_0)})$
$u(0,t) = 0$	—	$\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) - \mathcal{N}(-x_0, \sqrt{2D(t-t_0)})$
$\frac{\partial u}{\partial x}(0,t) = 0$	—	$\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) + \mathcal{N}(-x_0, \sqrt{2D(t-t_0)})$
$u(0,t) = 0$	$u(L,t) = 0$	$\sum_{k=1}^{\infty} \frac{2}{L}\ \sin(\frac{k\pi}{L} x_0)\ \sin(\frac{k\pi}{L} x)\ \exp(-(\frac{k\pi}{L})^2 D(t-t_0))$
$u(0,t) = 0$	$\frac{\partial u}{\partial x}(L,t) = 0$	$\sum_{k=0}^{\infty} \frac{2}{L}\ \sin(\frac{(k+\frac{1}{2})\pi}{L} x_0)\ \sin(\frac{(k+\frac{1}{2})\pi}{L} x)\ \exp(-(\frac{(k+\frac{1}{2})\pi}{L})^2 D(t-t_0))$
$\frac{\partial u}{\partial x}(0,t) = 0$	$u(L,t) = 0$	$\sum_{k=0}^{\infty} \frac{2}{L}\ \cos(\frac{(k+\frac{1}{2})\pi}{L} x_0)\ \cos(\frac{(k+\frac{1}{2})\pi}{L} x)\ \exp(-(\frac{(k+\frac{1}{2})\pi}{L})^2 D(t-t_0))$
$\frac{\partial u}{\partial x}(0,t) = 0$	$\frac{\partial u}{\partial x}(L,t) = 0$	$\frac{1}{L} + \sum_{k=1}^{\infty} \frac{2}{L}\ \cos(\frac{k\pi}{L} x_0)\ \cos(\frac{k\pi}{L} x)\ \exp(-(\frac{k\pi}{L})^2 D(t-t_0))$

Problem 1 (20 points): Short answer problems. Provide brief explanations (no lengthy derivations!) for each problem.

1. (5 points): Among all the eigenvalue-eigenvector pairs obtained by singular value decomposition of a matrix containing multi-dimensional data samples, which one explains most of the variance in the data?

The one with the largest eigenvalue (principal component).

2. (5 points): Give an example that demonstrates why diffusion over a bounded interval is not space-invariant.

A source acting on a zero-value boundary produces no response, whereas a source acting away from the boundary does not.

3. (5 points): To arrive at the diffusion transfer function $H(k, s)$, does it matter in which order the Fourier and Laplace transforms are applied to the diffusion equation, and why?

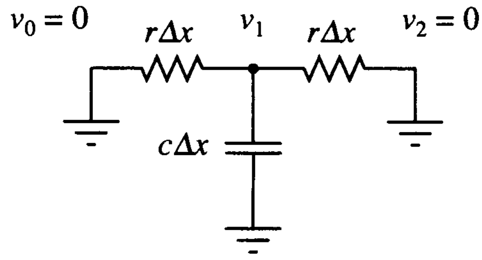
No: Fourier and Laplace are linear transforms, so the order in which they are applied doesn't matter.

4. (5 points): Solve the following integral:

$$\int_{-\infty}^{+\infty} \delta(x - x_0) \exp(-jkx)\, dx = \exp(-jkx_0)$$

Delta-Dirac picks the integrand at its center, x_0.

Problem 2 (30 points): Consider a two-segment lumped model of diffusion along a passive cable of length L, with line resistivity r and line capacitance c, and with zero-voltage boundary conditions on both ends, as shown below. The length of each of the two segments is $\Delta x = L/2$.



1. (10 points): Write the ordinary differential equation governing the dynamics of the voltage $v_1(t)$ at the center of the cable. Is this ODE homogeneous or inhomogeneous?

KCL @ v_1 node :

$$c\Delta x \frac{dv_1}{dt} = -\frac{1}{r\Delta x} \cdot v_1 - \frac{1}{r\Delta x} \cdot v_1$$

$$\Rightarrow \frac{dv_1}{dt} = -\frac{2}{rc\,\Delta x^2} \cdot v_1 = -\frac{8}{rc\,L^2} \cdot v_1$$

$$\Rightarrow \frac{dv_1}{dt} = -\frac{8D}{L^2} \cdot v_1 \quad \text{with} \quad D = \frac{1}{rc} \text{ diffusivity}$$

HOMOGENEOUS (zero source)

2. (10 points): Show that the solution to this ODE is given by a decaying exponential over time, $v_1(t) = A\exp(-t/\tau)$. Identify the amplitude constant A and the time constant τ in terms of the initial condition $v_1(0)$, cable length L, and diffusivity D.

$$v_1(t) = v_1(0) \cdot e^{-\frac{8D}{L^2}t}$$

$$= A\, e^{-\frac{1}{\tau}t}$$

with $\begin{cases} A = v_1(0) \\ \tau = \dfrac{L^2}{8D} \end{cases}$

3. (10 points): Compare the time constant τ that you obtained in part 2 for this two-segment lumped approximation of the passive cable, with the time constant τ_∞ for the infinite-segment continuum limit, corresponding to the dominant (first) term in the eigenmode series of the homogeneous solution for the cable with same diffusivity D, same total length L, and same zero-voltage boundary conditions on both ends.

First term: $v(x,t) = A_1 \sin\left(\frac{\pi x}{L}\right) e^{-\left(\frac{\pi}{L}\right)^2 D t}$

$$\Rightarrow \tau_\infty = \frac{L^2}{\pi^2 D}$$

Slightly (~15%) faster than τ

Problem 3 (50 points): Here we will consider diffusion of estrogen through tissue into the bloodstream. The diffusivity D is uniform across the tissue spanning length L from the skin at $x = 0$, to the vasculature at $x = L$. The skin is completely impermeable to estrogen at $x = 0$. The vasculature completely absorbs any estrogen at $x = L$. A subcutaneous patch implanted at $x = x_0$ continuously supplies a constant but infinitely concentrated source of estrogen into the tissue: $f(x,t) = q_0\,\delta(x - x_0)$ where q_0 is a constant, and $\delta(\cdot)$ is the delta-Dirac function. Initially the estrogen concentration $u(x,t)$ is given by:

$$u(x,0) = g(x) = \begin{cases} \frac{q_0}{D}(L - x_0) & \text{for} \quad 0 \le x \le x_0 \\ \frac{q_0}{D}(L - x) & \text{for} \quad x_0 < x \le L \end{cases}$$

1. (5 points): Write the partial differential equation governing the estrogen concentration $u(x,t)$ in the tissue. Express initial and boundary conditions.

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x,t) \; ; \quad \begin{cases} u(x,0) = g(x) & \text{I.C.} \\ \frac{\partial u}{\partial x}(0,t) = 0 & \text{FLUX B.C. @ } 0 \\ u(L,t) = 0 & \text{VALUE B.C. @ } L \end{cases}$$

with the given source $f(x,t)$ and I.C. $g(x)$

2. (15 points): Solve for a particular solution $u_p(x)$ for the estrogen concentration in the tissue at steady state.

$$0 = D\frac{d^2 u_p}{dx^2} + q_0\,\delta(x - x_0) \qquad \text{independent of time}$$

$$\Rightarrow \frac{d^2 u_p}{dx^2} = -\frac{q_0}{D}\,\delta(x - x_0)$$

$$\Rightarrow \frac{du_p}{dx} = -\frac{q_0}{D}\int \delta(x - x_0)\,dx = \begin{cases} C & \text{for } 0 \le x < x_0 \\ C - \frac{q_0}{D} & \text{for } x_0 < x \le L \end{cases}$$

FLUX B.C. @ $x = 0$: $\quad C = 0$

$$\Rightarrow u_p(x) = \int \frac{du_p}{dx}\,dx = \begin{cases} C' & \text{for } 0 \le x \le x_0 \\ C' - \frac{q_0}{D}(x - x_0) & \text{for } x_0 \le x \le L \end{cases}$$

VALUE B.C @ L ; $\quad c' - \frac{q_0}{D}(L - x_0) = 0$

$$\Rightarrow \quad u_p(x) = \begin{cases} c' = \frac{q_0}{D}(L - x_0) & \text{for } 0 \leq x \leq x_0 \\ c' - \frac{q_0}{D}(x - x_0) = \frac{q_0}{D}(L - x) & \text{for } x_0 \leq x \leq L \end{cases}$$

$$= g(x) \; !$$

3. (5 points): Write the homogeneous problem, with homogeneous partial differential equation and boundary conditions.

$$\frac{\partial u_H}{\partial t} = D \frac{\partial^2 u_H}{\partial x^2} \quad ; \quad \begin{cases} \frac{\partial u_H}{\partial x}(0,t) = 0 & \text{FLUX B.C @ 0} \\ u_H(L,t) = 0 & \text{VALUE B.C. @ L} \end{cases}$$

4. (10 points): Find the eigenmode decomposition for the general solution of the homogenous problem.

$$u_H(x,t) = \sum_{k=0}^{\infty} A_k \, \Phi_k(x) \, e^{-D\lambda_k t}$$

$$\begin{cases} \text{FLUX B.C @ 0} \\ \text{VALUE B.C @ L} \end{cases} : \qquad \lambda_k = \left(\frac{(k + \frac{1}{2})\pi}{L} \right)^2$$

$$\Phi_k(x) = \cos\left(\frac{(k + \frac{1}{2})\pi x}{L} \right)$$

5. (15 points): Solve for the estrogen concentration in the tissue over time $u(x,t)$ from the given initial conditions.

   Hint: KISS (Keep It Simple & Stupid). Don't get alarmed if your answer appears too simple. But do get alarmed if you're wielding through pages of integrals...

$$u(x,t) = u_p(x) + u_H(x,t)$$

$$= u_p(x) + \sum_{k=0}^{\infty} A_k \Phi_k(x) e^{-D\lambda_k t}$$

I.C.: $$g(x) = u(x,0) = u_p(x) + \sum_{k=0}^{\infty} A_k \Phi_k(x)$$

A_k: $$\sum_{k=0}^{\infty} A_k \Phi_k(x) = g(x) - u_p(x)$$

$$= 0$$

$$\Rightarrow A_k = 0 \text{ for all } k$$

$$\Rightarrow u(x,t) = u_p(x)$$