BENG 221 Mathematical Methods in Bioengineering

Fall 2017

Midterm

NAME: SOLUTIONS

- Open book, open notes.
- 80 minutes limit (end of class).
- No communication other than with instructor and TAs.
- No computers or internet, except for access to posted class materials.
### Table 1: Laplace and Fourier Transforms

<table>
<thead>
<tr>
<th>$u(t)$</th>
<th>$U(s)$</th>
<th>$u(t)$</th>
<th>$U(j\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
<td>$1$</td>
<td>$\frac{1}{j\omega}$</td>
</tr>
<tr>
<td>$e^{-at}$</td>
<td>$\frac{1}{s+a}$</td>
<td>$e^{-at}$ for $t \geq 0$; 0 otherwise</td>
<td>$\frac{1}{j\omega+a}$</td>
</tr>
<tr>
<td>$u(t-t_0)$; $t_0 \geq 0$</td>
<td>$\frac{1}{s}U(s)$</td>
<td>$u(t-t_0)$</td>
<td>$\frac{1}{j\omega}U(j\omega)$</td>
</tr>
<tr>
<td>$\frac{du}{dt}$</td>
<td>$sU(s) - u(0)$</td>
<td>$\frac{du}{dt}$</td>
<td>$j\omega U(j\omega)$</td>
</tr>
<tr>
<td>$\int_0^t u(t_0),dt_0$</td>
<td>$\frac{1}{s}U(s)$</td>
<td>$\int_0^t u(t_0),dt_0$</td>
<td>$\frac{1}{j\omega}U(j\omega)$</td>
</tr>
<tr>
<td>$\int_0^t f(t_0) h(t-t_0),dt_0$</td>
<td>$H(s) \cdot F(s)$</td>
<td>$\int_{-\infty}^t f(t_0) h(t-t_0),dt_0$</td>
<td>$H(j\omega) \cdot F(j\omega)$</td>
</tr>
</tbody>
</table>

### Table 2: Green’s Functions for Diffusion in 1-D

<table>
<thead>
<tr>
<th>B.C.</th>
<th>$G(x, t; x_0, t_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$u(0, t) = 0$</td>
</tr>
<tr>
<td>$x = L$</td>
<td>$u(L, t) = 0$</td>
</tr>
<tr>
<td></td>
<td>$\frac{\partial u}{\partial x}(0, t) = 0$</td>
</tr>
<tr>
<td></td>
<td>$\frac{\partial u}{\partial x}(L, t) = 0$</td>
</tr>
</tbody>
</table>

$$N(x_0, \sqrt{2D(t-t_0)}) = \frac{1}{\sqrt{4\pi D(t-t_0)}} \exp\left(-\frac{(x-x_0)^2}{4D(t-t_0)}\right)$$

$$\sum_{k=0}^{\infty} \frac{2}{L} \sin\left(\frac{k\pi}{L}x_0\right) \sin\left(\frac{k\pi}{L}x\right) \exp\left(-\frac{(k+\frac{1}{2})\pi}{L}D(t-t_0)\right)$$

$$\sum_{k=0}^{\infty} \frac{2}{L} \cos\left(\frac{k+\frac{1}{2}}{L}x_0\right) \cos\left(\frac{k+\frac{1}{2}}{L}x\right) \exp\left(-\left(\frac{(k+\frac{1}{2})\pi}{L}D(t-t_0)\right)^2\right)$$
Problem 1 (20 points): Short answer problems. Provide brief explanations (no lengthy derivations!) for each problem.

1. (5 points): Among all the eigenvalue-eigenvector pairs obtained by singular value decomposition of a matrix containing multi-dimensional data samples, which one explains most of the variance in the data?

   The one with the largest eigenvalue (principal component).

2. (5 points): Give an example that demonstrates why diffusion over a bounded interval is not space-invariant.

   A source acting on a zero-value boundary produces no response, whereas a source acting away from the boundary does not.

3. (5 points): To arrive at the diffusion transfer function $H(k, s)$, does it matter in which order the Fourier and Laplace transforms are applied to the diffusion equation, and why?

   No: Fourier and Laplace are linear transforms, so the order in which they are applied doesn’t matter.

4. (5 points): Solve the following integral:

   $$\int_{-\infty}^{+\infty} \delta(x - x_0) \exp(-jkx) \, dx = \exp(-jkx_0)$$

   Delta-Dirac picks the integrand at its center, $x_0$. 

Problem 2  (30 points): Consider a two-segment lumped model of diffusion along a passive cable of length \( L \), with line resistivity \( r \) and line capacitance \( c \), and with zero-voltage boundary conditions on both ends, as shown below. The length of each of the two segments is \( \Delta x = L/2 \).

![Diagram](image)

1. (10 points): Write the ordinary differential equation governing the dynamics of the voltage \( v_1(t) \) at the center of the cable. Is this ODE homogeneous or inhomogeneous?

\[
\begin{align*}
\text{KCL @ } v_1 \text{ node:} & \\
& c \Delta x \frac{dv_1}{dt} = - \frac{1}{r \Delta x} \cdot v_1 - \frac{1}{r \Delta x} \cdot v_1 \\
\Rightarrow & \quad \frac{dv_1}{dt} = - \frac{2}{r c \Delta x^2} \cdot v_1 = - \frac{8}{r c L^2} \cdot v_1 \\
\Rightarrow & \quad \frac{dv_1}{dt} = - \frac{8 D}{L^2} \cdot v_1 \quad \text{with} \quad D = \frac{1}{r c} \quad \text{diffusivity} \\
\text{Homogeneous (zero source)}
\end{align*}
\]
2. (10 points): Show that the solution to this ODE is given by a decaying exponential over time, \( v_1(t) = A \exp(-t/\tau) \). Identify the amplitude constant \( A \) and the time constant \( \tau \) in terms of the initial condition \( v_1(0) \), cable length \( L \), and diffusivity \( D \).

\[
\begin{align*}
v_1(t) &= v_1(0) \cdot e^{-\frac{8D}{L^2} t} \\
&= A \cdot e^{-\frac{t}{\tau}}
\end{align*}
\]

with
\[
\begin{align*}
A &= v_1(0) \\
\tau &= \frac{L^2}{8D}
\end{align*}
\]

3. (10 points): Compare the time constant \( \tau \) that you obtained in part 2 for this two-segment lumped approximation of the passive cable, with the time constant \( \tau_\infty \) for the infinite-segment continuum limit, corresponding to the dominant (first) term in the eigenmode series of the homogeneous solution for the cable with same diffusivity \( D \), same total length \( L \), and same zero-voltage boundary conditions on both ends.

First term: \( v(x,t) = A_1 \sin\left(\frac{\pi x}{L}\right) e^{-\left(\frac{\pi^2}{L^2}\right)D t} \)

\[
\Rightarrow \quad \tau_\infty = \frac{L^2}{\pi^2 D}
\]

Slightly (~15%) faster than \( \tau \)
Problem 3 (50 points): Here we will consider diffusion of estrogen through tissue into the bloodstream. The diffusivity $D$ is uniform across the tissue spanning length $L$ from the skin at $x = 0$, to the vasculature at $x = L$. The skin is completely impermeable to estrogen at $x = 0$. The vasculature completely absorbs any estrogen at $x = L$. A subcutaneous patch implanted at $x = x_0$ continuously supplies a constant but infinitely concentrated source of estrogen into the tissue: $f(x, t) = q_0 \delta(x - x_0)$ where $q_0$ is a constant, and $\delta(\cdot)$ is the delta-Dirac function. Initially the estrogen concentration $u(x, t)$ is given by:

$$u(x, 0) = g(x) = \begin{cases} \frac{q_0}{D} (L - x_0) & \text{for } 0 \leq x \leq x_0 \\ \frac{q_0}{D} (L - x) & \text{for } x_0 < x \leq L \end{cases}$$

1. (5 points): Write the partial differential equation governing the estrogen concentration $u(x, t)$ in the tissue. Express initial and boundary conditions.

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(x, t) \quad \text{I.C.} \quad \begin{cases} u(x, 0) = g(x) \\ \frac{\partial u}{\partial x}(0, t) = 0 \quad \text{flux b.c. at } 0 \\ u(L, t) = 0 \quad \text{value b.c. at } L \end{cases}$$

with the given source $f(x, t)$ and I.C. $g(x)$

2. (15 points): Solve for a particular solution $u_p(x)$ for the estrogen concentration in the tissue at steady state.

$$0 = D \frac{d^2 u_p}{dx^2} + q_0 \delta(x-x_0) \quad \text{independent of time}$$

$$\Rightarrow \frac{d^2 u_p}{dx^2} = -\frac{q_0}{D} \delta(x-x_0)$$

$$\Rightarrow \frac{du_p}{dx} = -\frac{q_0}{D} \int \delta(x-x_0) dx = \begin{cases} c \\ c - \frac{q_0}{D} \end{cases} \begin{cases} \text{for } 0 \leq x < x_0 \\ \text{for } x_0 < x \leq L \end{cases}$$

**Flux B.C. @ x=0:** \quad C = 0

$$\Rightarrow u_p(x) = \int \frac{du_p}{dx} dx = \begin{cases} c^2 \\ c^2 - \frac{q_0}{D} (x-x_0) \end{cases} \begin{cases} \text{for } 0 \leq x \leq x_0 \\ \text{for } x_0 \leq x \leq L \end{cases}$$
\[
\text{VALUE BC @ } L : \quad c' - \frac{q_o}{D} (L-x_0) = 0
\]

\[
\therefore \quad M_p(x) = \begin{cases} 
  c' = \frac{q_o}{D} (L-x_0) & \text{for } 0 \leq x_0, \quad 0 \leq x < x_0 \\
  c' - \frac{q_o}{D} (x-x_0) = \frac{q_o}{D} (L-x) & \text{for } x_0 \leq x \leq L 
\end{cases}
\]

\[
= g(x) !
\]

3. (5 points): Write the homogeneous problem, with homogeneous partial differential equation and boundary conditions.

\[
\frac{\partial^2 m_H}{\partial t^2} = D \frac{\partial^2 m_H}{\partial x^2} ; \quad \left\{ \begin{array}{l}
\frac{\partial m_H(0, t)}{\partial x} = 0 \quad \text{FLUX BC @ 0} \\
 m_H(L, t) = 0 \quad \text{VALUE BC @ L}
\end{array} \right.
\]

4. (10 points): Find the eigenmode decomposition for the general solution of the homogeneous problem.

\[
m_H(x, t) = \sum_{k=0}^{\infty} A_k \Phi_k(x) e^{-D\lambda_k t}
\]

\[
\left\{ \begin{array}{l}
\text{FLUX BC @ 0} \\
\text{VALUE BC @ L}
\end{array} \right. : \quad \lambda_k = \left( \frac{\left(\frac{k}{2} + \frac{1}{2}\right) \pi}{L} \right)^2
\]

\[
\Phi_k(x) = \cos \left( \frac{\left(\frac{k}{2} + \frac{1}{2}\right) \pi x}{L} \right)
\]
5. (15 points): Solve for the estrogen concentration in the tissue over time $u(x,t)$ from the given initial conditions.

*Hint:* KISS (Keep It Simple & Stupid). Don’t get alarmed if your answer appears too simple. But do get alarmed if you’re wielding through pages of integrals...

\[
m(x,t) = m_p(x) + m_h(x,t) = m_p(x) + \sum_{k=0}^{\infty} A_k \phi_k(x) e^{-D_k t}
\]

I.C.: \( g(x) = m(x,0) = m_p(x) + \sum_{k=0}^{\infty} A_k \phi_k(x) \)

\[
A_k: \sum_{k=0}^{\infty} A_k \phi_k(x) = g(x) - m_p(x) = 0
\]

\[\Rightarrow A_k = 0 \text{ for all } k\]

\[\Rightarrow m(x,t) = m_p(x)\]