BENG 221

DRUG DIFFUSION THROUGH A COATED STENT

12TH NOVEMBER, 2010

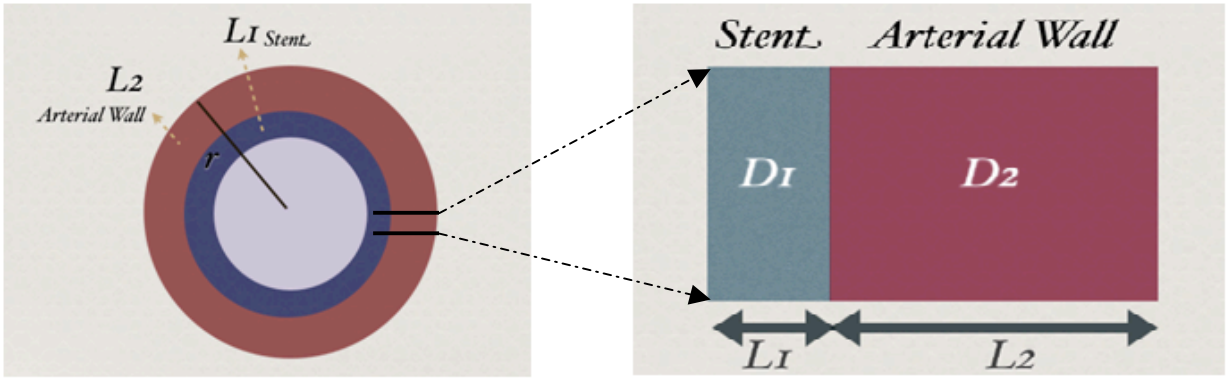
1. INTRODUCTION:

In this report, a bioengineering application of the one-dimensional diffusion equation, solved in both rectangular and cylindrical coordinates, is considered. An example of this is the diffusion of a drug through an arterial wall via a coated stent.

A stent is a metallic prosthesis implanted into the arterial wall and coated with a layer of a therapeutic drug. It is used to treat heart diseases such as atherosclerosis, which can be considered a form of chronic inflammation. When it affects coronary arteries, symptoms such as angina pectoris and heart attack can occur[1]. So as to revascularize coronary arteries, a stent can be used. A stent is an expandable metal or polymeric tubular mesh. A drug diffusing stent is a normal metal stent that has been coated with a pharmacologic agent (drug) that is known to interfere with the process of restenosis (reblocking).

In this problem, a drug-diffusing stent coated with heparin is considered. Once the stent is inserted into an artery, it starts diffusing through the arterial wall. The solution indicates the concentration of the drug at different distances in the wall, at different time intervals. This is important as it helps to determine the rate at which the drug is diffusing through the artery. Depending on the time at which all drug has diffused through the wall, we can determine if modifications in the system are required for sustained drug delivery.

2. SET-UP:



CASE II CASE I

Figure 2.1 Cross section of artery

a.) **CASE I**: Consider one portion of the arterial wall

We can make the following assumptions in this condition:

- As the vacant space in the artery is much larger than the thickness of either the stent or the wall, we can consider only one part of the blood vessel, and assume the stent and wall to be two parts of a slab. Considering this assumption, the problem can be solved in rectangular coordinates.
- The diffusivities of both the coated stent and the wall are constant.
- The stent is impervious to flow of blood (that is, the drug will only diffuse in one direction).
- The outer edge of the wall is impermeable to the drug.
- Differential equation: $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$
- Initial condition: C(x,0)=0; initially, the drug concentration in the wall is zero.
- Boundary condition:
    1. $C(0,t)=C_{max}=1$ mol/cm^3; the amount of drug on the surface of the stent is much higher than at any other point in the wall. So, the stent is a constant source of the drug over time.
    2. Dc/dt(L,t)=0; at the end of the arterial wall, the flux is zero due to impermeability.

This problem is first solved analytically considering the entire setup to be made up of one slab. Then, the problem is solved numerically considering two layers.

b.) **CASE II**: Consider the artery to be a cylinder. A stent is an expandable metal or polymeric tubular mesh. It is difficult to solve this problem with the stent as a mesh, as it is then not a uniform source of the drug. Certain assumptions are therefore made to modify our problem into a more convenient form, mathematically.

The assumptions made in this case are:

- The stent is assumed to be a film of uniform thickness coating the inner wall of the blood vessel.
- The artery is assumed to be a cylinder of uniform thickness.
- The diffusivities of both the coated stent and the wall are constant.
- The stent is impervious to flow of blood (that is, the drug will only diffuse in one direction).
- The outer wall of the blood vessel is impermeable to the drug.

- Differential equation: $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial C}{\partial r}) = \frac{1}{D}\frac{\partial C}{\partial t}$
- Initial condition: C(x,0)=0
- Boundary conditions:
  1. C(r,t)= C_{max}= 1 mol/cm^3,
  2. dc/dx(r2,t)=0

3. SOLUTION:

3.1 ANALYTICAL SOLUTION FOR CARTESIAN COORDINATE:

Diffusion Equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \quad (3.1.1)$$

Initial Condition

$$C(x,0) = 0 \quad (3.1.2)$$

Boundary Conditions

$$C(0,t) = C_{max} \quad (3.1.3)$$

$$\frac{\partial C}{\partial x}(L,t) = 0 \quad (3.1.4)$$

This is a diffusion problem in one-dimensional slab with constant physical properties and no sources. To solve this equation, solution can be separated into homogeneous and particular solutions. Particular solution is the steady state solution where λ=0.

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \quad (3.1.5)$$

$$\frac{\partial^2 C}{\partial x^2} = 0 \quad (3.1.6)$$

$$\frac{\partial C}{\partial x} = A_p \quad (3.1.7)$$

$$C_p(x,t) = A_p x + B_p \quad (3.1.8)$$

$$Applying\ boundary\ condition\ C(0,t) = C_{max}\ gives$$

$$C_p(0,t) = A_p(0) + B_p = C_{max} \quad (3.1.9)$$

$$B_p = C_{max} \quad (3.1.10)$$

Applying boundary condition $\frac{\partial C}{\partial x}(L,t) = 0$ *gives*

$$\frac{\partial C}{\partial x}(L,t) = A_p = 0 \tag{3.1.11}$$

Particular solution becomes

$$C_p(x,t) = C_{max} \tag{3.1.12}$$

Homogeneous solution can be found using separation of variables

$$Let\ C_h(x,t) = F(x)G(t) \tag{3.1.13}$$

Take derivative of C with respect to t

$$\frac{\partial C_h}{\partial t} = F\frac{\partial G}{\partial t} \tag{3.1.14}$$

Take second derivative of C with respect to x

$$\frac{\partial^2 C_h}{\partial x^2} = G\frac{\partial^2 F}{\partial x^2} \tag{3.1.15}$$

Substituting into diffusion equation gives

$$F\frac{\partial G}{\partial t} = DG\frac{\partial^2 F}{\partial x^2} \tag{3.1.16}$$

Equations are rearranged as

$$\frac{1}{DG}\frac{\partial G}{\partial t} = \frac{1}{F}\frac{\partial^2 F}{\partial x^2} \tag{3.1.17}$$

Both sides should be equal the same constant.

$$\frac{1}{DG}\frac{\partial G}{\partial t} = \frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\lambda^2 \tag{3.1.18}$$

Then, these two ODE can be solved separately.

$$\frac{1}{DG}\frac{\partial G}{\partial t} = -\lambda^2 \tag{3.1.19}$$

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\lambda^2 \tag{3.1.20}$$

Solving equation (3.1.19)

$$\frac{1}{DG}\frac{\partial G}{\partial t} = -\lambda^2 \tag{3.1.21}$$

$$\frac{\partial G}{\partial t} = -\lambda^2 DG \tag{3.1.22}$$

$$G(t) = c_0 e^{-\lambda^2 Dt} \tag{3.1.23}$$

Solving equation (3.1.20)

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\lambda^2 \quad (3.1.24)$$

$$\frac{\partial^2 F}{\partial x^2} = -\lambda^2 F \quad (3.1.25)$$

Characteristic equation of ODE is as follows:

$$r^2 + \lambda^2 = 0 \quad (3.1.26)$$

The physically relevant solution is found for $\lambda > 0$. *This gives*

$$r = \pm i\lambda \quad (3.1.27)$$

The form of the general solution is

$$F(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x} \quad (3.1.28)$$

$$F(x) = c_3 \cos(\lambda x) + c_4 \sin(\lambda x) \quad (3.1.29)$$

General homogeneous solution of $C(x,t)$

$$C_h(x,t) = G(t)F(x) \quad (3.1.30)$$

$$= c_0 e^{-\lambda^2 Dt}(c_3 \cos(\lambda x) + c_4 \sin(\lambda x)) \quad (3.1.31)$$

$$= e^{-\lambda^2 Dt}(A\cos(\lambda x) + B\sin(\lambda x)) \quad (3.1.32)$$

To find A and B, boundary conditions are applied. However, initial condition and boundary conditions for each solution (particular and homogeneous)should add up to the overall initial and boundary conditions. Thus, homogeneous boundary conditions should be found as:

$$C(0,t) = C_p(0,t) + C_h(0,t) \quad (3.1.33)$$

$$C_h(0,t) = C(0,t) - C_p(0,t) \quad (3.1.34)$$

$$C_h(0,t) = C_{max} - C_{max} = 0 \quad (3.1.35)$$

Applying this boundary condition gives

$$C_h(0,t) = e^{-\lambda^2 Dt}(A\cos(\lambda(0)) + B\sin(\lambda(0))) = 0 \quad (3.1.36)$$

$$A = 0 \quad (3.1.37)$$

$$C_h(x,t) = e^{-\lambda^2 Dt}(B\sin(\lambda x)) \quad (3.1.38)$$

Applying homogeneous boundary condition $\frac{\partial C_h}{\partial x}(L,t) = 0$ *gives*

$$\frac{\partial C_h}{\partial x} = e^{-\lambda^2 Dt}(\lambda B\cos(\lambda x)) \quad (3.1.39)$$

$$\frac{\partial C_h}{\partial x}(L,t) = e^{-\lambda^2 Dt}(\lambda B\cos(\lambda L)) = 0 \tag{3.1.40}$$

$$B\cos(\lambda L) = 0 \tag{3.1.41}$$

for non trivial solution, $B \neq 0$ *so* $\cos(\lambda L) = 0$

$$\lambda L = \frac{(2n-1)\pi}{2} \tag{3.1.42}$$

$$\lambda = \frac{(2n-1)\pi}{2L} \tag{3.1.43}$$

Solving for homogeneous solution of $C_h(x,t)$

$$C_h(x,t) = \sum_{n=1}^{n=\infty} B_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)\exp\left(-\left(\frac{(2n-1)\pi}{2L}\right)^2 Dt\right) \tag{3.1.44}$$

Apply homogenous initial condition $C(x,0) = C_p(x,0) + C_h(x,0)$ (3.1.45)

$$C_h(x,0) = C(x,0) - C_p(x,0) \tag{3.1.46}$$

$$C_h(x,0) = 0 - C_{max} = -C_{max} \tag{3.1.47}$$

$$C_h(x,0) = \sum_{n=1}^{n=\infty} B_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)\exp\left(-\left(\frac{(2n-1)\pi}{2L}\right)^2 D(0)\right) = -C_{max} \tag{3.1.48}$$

$$\sum_{n=1}^{n=\infty} B_n \sin\left(\frac{(2n-1)\pi}{2L}x\right) = -C_{max} \tag{3.1.49}$$

Using orthogonality of sines, $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0 \text{ when } m \neq n \\ \frac{L}{2} \text{ when } m = n \end{cases}$ (3.1.50)

$$\sum_{n=1}^{n=\infty} B_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)\sin\left(\frac{(2m-1)\pi}{2L}x\right) = -C_{max}\sin\left(\frac{(2m-1)\pi}{2L}x\right) \tag{3.1.51}$$

$$\sum_{n=1}^{n=\infty} B_n \int_0^L \sin\left(\frac{(2n-1)\pi}{2L}x\right)\sin\left(\frac{(2m-1)\pi}{2L}x\right)dx = -C_{max}\int_0^L \sin\left(\frac{(2m-1)\pi}{2L}x\right)dx \tag{3.1.52}$$

$$When\ m = n,\ \ B_n \int_0^L sin^2\left(\frac{n\pi x}{l}\right)dx = -C_{max}\left[-\frac{2L}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{2L}x\right)\right]_0^L \tag{3.1.53}$$

$$B_n\ \frac{L}{2} = -C_{max}\left[-\frac{2L}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{2L}x\right)\right]_0^L \tag{3.1.54}$$

$$B_n = \frac{4C_{max}}{(2n-1)\pi}\left[\cos\left(\frac{(2n-1)\pi}{2L}L\right) - \cos(0)\right] \tag{3.1.55}$$

$$B_n = \frac{4C_{max}}{(2n-1)\pi}\left[\cos\left(\frac{(2n-1)\pi}{2}\right) - 1\right] \tag{3.1.56}$$

$$B_n = \frac{4C_{max}}{(2n-1)\pi}[0-1] \tag{3.1.57}$$

$$B_n = -\frac{4C_{max}}{(2n-1)\pi} \tag{3.1.58}$$

Homogeneous solution becomes

$$C_h(x,t) = \sum_{n=1}^{n=\infty} -\frac{4C_{max}}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi}{2L}x\right)\exp\left(-\left(\frac{(2n-1)\pi}{2L}\right)^2 Dt\right) \tag{3.1.59}$$

Final solution is found by adding particular and homogeneous solutions:

$$C(x,t) = C_p(x,t) + C_h(x,t) \tag{3.1.60}$$

$$C(x,t) = C_{max} - \sum_{n=1}^{n=\infty} \frac{4C_{max}}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2L}x\right) \exp\left(-\left(\frac{(2n-1)\pi}{2L}\right)^2 Dt\right) \tag{3.1.61}$$

3.2. ANALYTICAL SOLUTION FOR CYLINDRICAL COORDINATE:

Diffusion Equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) = \frac{1}{D}\frac{\partial C}{\partial t} \tag{3.2.1}$$

Initial Condition

$$C(x,0) = 0 \tag{3.2.2}$$

Boundary Conditions

$$C(r_1,t) = C_{\max} \tag{3.2.3}$$

$$\frac{\partial C}{\partial x}(r_2,t) = 0 \tag{3.2.4}$$

This is a diffusion problem in r-direction with constant physical properties and no sources. To solve this equation, solution can be separated into homogeneous and particular solutions. Particular solution is the steady state solution where λ=0.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dC}{dr}\right) = 0 \tag{3.2.5}$$

$$r\frac{dC}{dr} = A \tag{3.2.6}$$

$$\int dC = \int \frac{A}{r} dr \tag{3.2.7}$$

$$C_p(x,t) = A\ln r + B \tag{3.2.8}$$

At r=0, $C_p(x,t)$ should be finite. Since ln(0) is not a finite value:

A=0 (3.2.9)

$$C_p(x,t) = B \tag{3.2.10}$$

Applying boundary condition $C(r_2,t)=C_{max}$ gives

$$C_{max}=B \tag{3.2.11}$$

Particular solution becomes

$$C_p(x,t) = C_{\max} \tag{3.2.12}$$

Homogeneous solution can be found using separation of variables

$$C_h(r,t) = G(r)F(t) \tag{3.2.13}$$

Substituting into diffusion equation

$$\frac{1}{r}(\frac{d}{dr}(rF\frac{dG}{dr})) = \frac{G}{D}\frac{dF}{dt} \tag{3.2.14}$$

$$\frac{1}{G*r}\frac{dG}{dr} + \frac{1}{G}\frac{d^2G}{dr^2} = \frac{1}{F*D}\frac{dF}{dt} \tag{3.2.15}$$

Both sides should be equal the same constant.

$$\frac{1}{G*r}\frac{dG}{dr} + \frac{1}{G}\frac{d^2G}{dr^2} = \frac{1}{F*D}\frac{dF}{dt} = -\lambda^2 \tag{3.2.16}$$

where λ is greater than 0.

Then, these two ODE can be solved separately.

$$\frac{1}{F*D}\frac{dF}{dt} = -\lambda^2 \tag{3.2.17}$$

$$\frac{1}{G*r}\frac{dG}{dr} + \frac{1}{G}\frac{d^2G}{dr^2} = -\lambda^2 \tag{3.2.18}$$

Solving equation (3.2.17)

$$\frac{1}{F*D}\frac{dF}{dt} = -\lambda^2 \tag{3.2.19}$$

$$\frac{dF}{F} = -\int \lambda^2 D dt \tag{3.2.20}$$

$$\ln F = -\lambda^2 Dt + K \tag{3.2.21}$$

$$F = F_0 e^{-\lambda^2 Dt} \tag{3.2.22}$$

Solving equation (3.2.18)

$$\frac{1}{G*r}\frac{dG}{dr} + \frac{1}{G}\frac{d^2G}{dr^2} = -\lambda^2 \tag{3.2.23}$$

Multiplying by r^2 and G gives

$$r^2 \frac{d^2G}{dr^2} + r\frac{dG}{dr} + Gr^2\lambda^2 = 0 \tag{3.2.24}$$

The general solution of any linear homogeneous second-order differential equation is a linear combination of two independent solutions. Thus, the general solution of Bessel's differential equation:

$$G(r) = A * J_0(\lambda r) + B * Y_0(\lambda r) \tag{3.2.25}$$

At r=0, C should be finite. Since $Y_0(0) = \infty$

B=0 (3.2.26)

Applying the second homogeneous boundary condition

$$\frac{\partial}{\partial r}(C(r_2,t)) = \frac{d}{dr}(AJ_0(\lambda r)) = -\lambda A J_1(\lambda r_2) = 0 \tag{3.2.27}$$

There are multiple eigenvalues that make $J_1(\lambda r_2)$ zero. Let's call $\lambda=\lambda_n$

$$G = A * J_0(\lambda_n r) \tag{3.2.28}$$

Solving for homogeneous solution of $C_h(x,t)$

$$C_h(x,t) = G * F = \sum_{n=1}^{\infty} C_n e^{-D\lambda_n^2 t} J_0(\lambda_n r) \tag{3.2.29}$$

where $C_n = AxF_0$

C_n can be found by applying homogeneous initial condition.

$$C(x,t) = C_h(x,t) + C_p(x,t) \tag{3.2.30}$$

$$C_h(x,0) = C(x,0) - C_p(x,0) = 0 - C_{\max} = -C_{\max} \tag{3.2.31}$$

The Fourier-Bessel representation of f(r):

$$C_h(x,0) = -C_{\max} = f(r) = \sum_{n=1}^{\infty} C_n J_0(\lambda_n r) \tag{3.2.32}$$

Using the orthogonality property of the eigenfunctions:

$$\int_{r_1}^{r_2} rJ_0(\lambda_m r) f(r) dr = \int_{r_1}^{r_2} C_n J_0(\lambda_n r) rJ_0(\lambda_m r) dr \tag{3.2.33}$$

$$\int_{r_1}^{r_2} rJ_0(\lambda_m r) f(r) dr = C_n \left(\frac{r_2^2}{2}(J_0^2(\lambda_m r_2) + J_1^2(\lambda_m r_2)) - \frac{r_1^2}{2}(J_0^2(\lambda_m r_1) + J_1^2(\lambda_m r_1))\right) \tag{3.2.34}$$

$$\int_{r_1}^{r_2} rJ_0(\lambda_m r) f(r) dr = C_n \left(\frac{r_2^2}{2}(J_0^2(\lambda_m r_2) - \frac{r_1^2}{2}(J_0^2(\lambda_m r_1) + J_1^2(\lambda_m r_1))\right) \tag{3.2.35}$$

$$C_n = \frac{2}{r_2^2 J_0^2(\lambda_m r_2) - r_1^2 (J_1^2(\lambda_m r_1) + J_0^2(\lambda_m r_1))} \int_{r_1}^{r_2} rJ_0(\lambda_m r) f(r) dr \tag{3.2.36}$$

The right hand side of the equation can be found as:

$$\int_{r_1}^{r_2} rJ_0(\lambda_m r) f(r) dr = -\frac{f(r)}{\lambda_m}(r_2 J_1(\lambda_m r_2) - r_1 J_1(\lambda_m r_1)) \tag{3.2.37}$$

where f(r)=constant

$$C_n = \frac{2C_{\max}}{r_2^2 J_0^2(\lambda_m r_2) - r_1^2 (J_1^2(\lambda_m r_1) + J_0^2(\lambda_m r_1))} \left(\frac{r_2 J_1(\lambda_m r_2) - r_1 J_1(\lambda_m r_1)}{\lambda_m}\right) \tag{3.2.38}$$

Due to the eigenvalues, $J_1(\lambda_m r_2)=0$ (3.2.39)

$$C_n = \frac{-2r_1 J_1(\lambda_m r_1) C_{\max}}{\lambda_m (r_2^2 J_0^2(\lambda_m r_2) - r_1^2 (J_1^2(\lambda_m r_1) + J_0^2(\lambda_m r_1)))} \tag{3.2.40}$$

Homogeneous solution becomes

$$C_h(x,t) = G * F = \sum_{n=1}^{\infty} C_n e^{-D\lambda_n^2 t} J_0(\lambda_n r) \tag{3.2.41}$$

Final solution is found by adding particular and homogeneous solutions:

$$C(x,t) = C_h(x,t) + C_p(x,t) \tag{3.2.42}$$

$$C(x,t) = C_h(x,t) + C_p(x,t) = \sum_{n=1}^{\infty} \frac{-2r_1 J_1(\lambda_n r_1) C_{\max} e^{-D\lambda_n t} J_0(\lambda_n r)}{\lambda_m (r_2^2 J_0^2(\lambda_n r_2) - r_1^2 (J_1^2(\lambda_n r_1) + J_0^2(\lambda_n r_1)))} + C_{\max} \tag{3.2.43}$$

3.3. NUMERICAL ANALYSIS IN CARTESIAN COORDINATES:

Starting from the original diffusion equation:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \tag{3.3.1}$$

we need to write our equation in a form that matlab can solve it:

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right) \tag{3.3.2}$$

for Cartesian coordinates m = 0, so with simplification we have:

$$c * \frac{\partial u}{\partial t} = \frac{\partial f}{\partial x} + s \tag{3.3.3}$$

comparing equations 3.1 and 3.3 we have:

c=1

$$f(x,t,u,\frac{\partial u}{\partial x}) = D * \frac{\partial u}{\partial x} \tag{3.3.4}$$

s=0

The solution needs to satisfy the initial condition u(x,0) = 0, so we have the term u0 = 0 which is identified in the icpde() in the matlab code.

Also the solution needs to satisfy the boundary conditions xl=0 and xr = L1+L2

$$p(x, t, u) + q(x, t) f\left(x, t, u, \frac{\partial u}{\partial x}\right) = 0 \qquad (3.3.5)$$

We already have f found in equation 3.4, so we have:

p + q *D* $\frac{\partial u}{\partial x}$ = 0

For the left boundary, u(0,t) = Cmax, so u(0,t)-Cmax = 0

ul – Cmax = 0. As a result, pl = ul – Cmax and ql=0 , since we don't have any $\frac{\partial u}{\partial x}$ term.

For the right boundary, we assumed the flux in the outside wall of the blood vessel to be zero:

$\frac{\partial u}{\partial x}$=0 , so we can write:

0 + 1 * $\frac{\partial u}{\partial x}$ = 0 (3.3.6)

as a result, pr = 0 and qr=1.

3.4.NUMERICAL ANALYSIS IN CYLINDRICAL COORDINATES:

In cylindrical coordinates, all the initial and boundary conditions are the same. The only term which varies is m=1.

Matlab results (Plot) for Cartesian Coordinates:

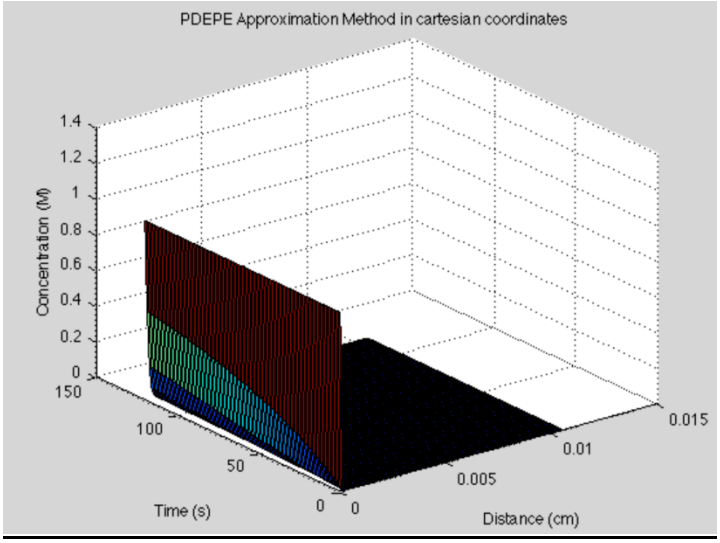


Figure 3.3.1 Surface plot of PDEPE numerical analysis in Cartesian coordinates.

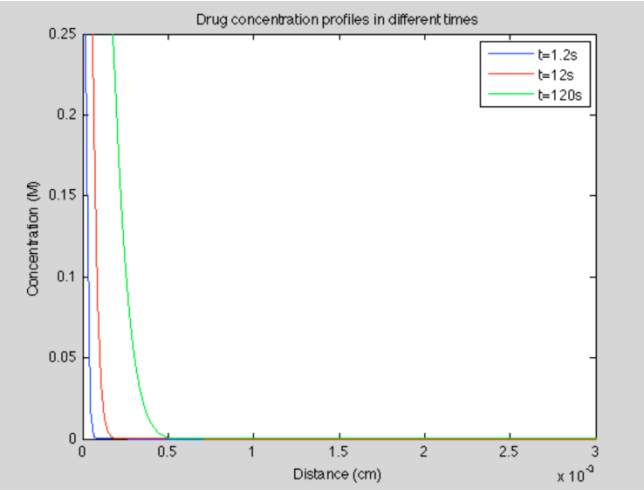


Figure 3.3.2 Drug concentration profile in different times in Cartesian coordinates

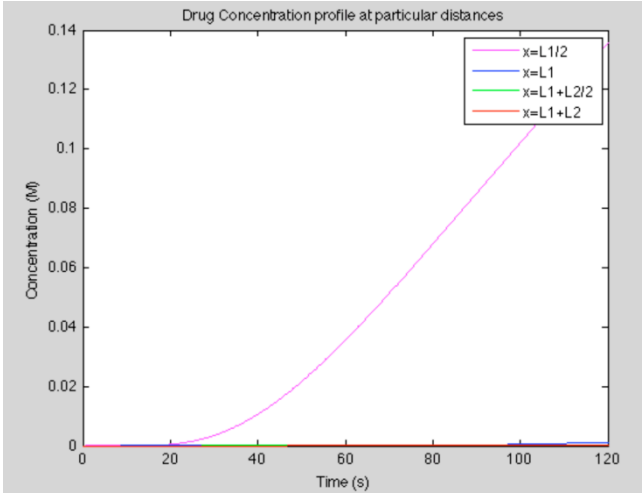


Figure 3.3.3 Drug Concentration profile in different distances in Cartesian coordinates

Observation: The drug concentration suddenly drops after a very short distance from the stent wall. Figure 3.3.3 shows that the concentration in certain distance (L1/2) in the middle of the stent is increased overtime, which is reasonable but the concentrations in distances further from the origin all

fall into the zero level and barely increase. This indicates that the drug diffusion needs to be much slower.

Matlab results (Plot) for Cylindrical Coordinates:

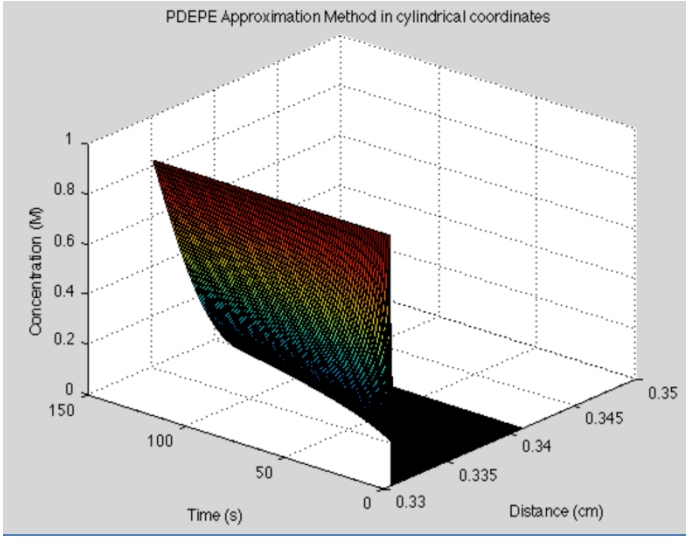


Figure 3.3.4 Surface plot for PDEPE numerical analysis in cylindrical coordinates

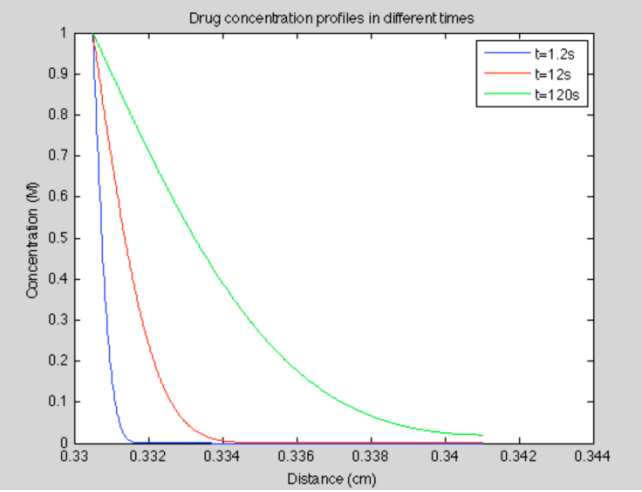


Figure 3.3.5 rug concentration profile in different times in cylindrical coordinates

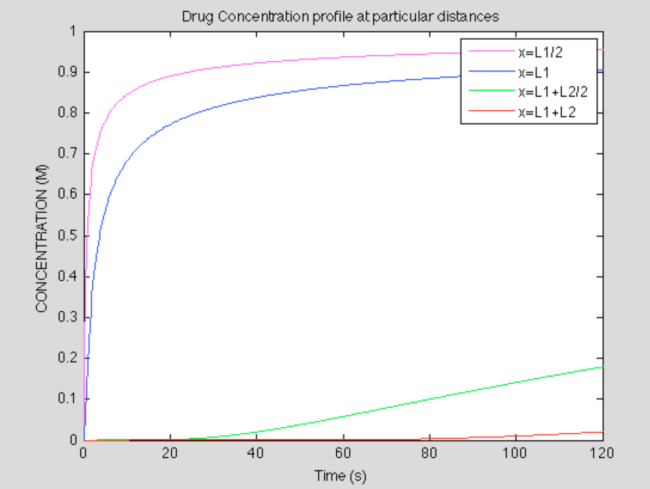


Figure 3.3.6 Drug Concentration profile in different distances in cylindrical coordinates

Observation: Using cylindrical coordinates, the plots demonstrate more realistic conditions. The surface plot shows a more gradual change in the drug concentration over time and distance. As a result, compared to the Cartesian coordinates, the concentration drops to zero at a farther distance away from the origin. Also, in figure 3.3.6, the concentration increase over time, approaching the maximum concentration, is more visible.

Matlab Code (Cartesian Coordinates Method):

This code presents numerical analysis of diffusion equation in Cartesian coordinates using MATLAB PDEPE method.

```
function presentation()

%define constants taken from the research paper[2][3]

 L1 = 5e-4; % stent thickness in cm

 L2 = 0.01; % artery wall thickness in cm

 x = linspace( 0, L1+L2, 500); % distance variable starting from inside radius going to outside radius

 t = linspace(0,120,500); % time variable in s

 % pdepe method in cylindrical coordinates

sol = pdepe(0,@mypde,@icpde,@bcpde,x,t);

% Extract the first solution component as C.

C = sol(:,:,1);

% A surface plot representing the concentration vs. distance and time

figure(1)

surf(x,t,C);

title('PDEPE Approximation Method in cartesian coordinates');

xlabel('Distance (cm)'), ylabel('Time (s)'), zlabel('Concentration (M)');

% A solution profile observing concentration change in certain distances

% over time

figure(2)

plot(t,C(:,12),'m')

hold on

plot(t,C(:,24),'b')

hold on

plot(t,C(:,262),'g')
```

```
hold on
plot(t,C(:,499),'r')
title('Drug Concentration profile at particular distances');
legend('x=L1/2','x=L1','x=L1+L2/2','x=L1+L2');
xlabel('Time (s)'), ylabel('Concentration (M)');
% A solution profile observing concentration change in certain times
% over distance
figure (3)
plot (x, C(5,:),'b')
hold on
plot (x, C(50,:), 'r')
hold on
plot (x, C(499,:), 'g')
title('Drug concentration profiles in different times');
legend ('t=1.2s', 't=12s','t=120s');
xlabel ('Distance (cm)'), ylabel('Concentration (M)');
 % function to define the components of pde as Matlab can recognize
% from the general form D*d2C/dx2 = dC/dt
% find c, f, s such that our equation fits c(x,t,C,DC/Dx) * DC/Dt = x^(-m)
% * D(x^m * f(x,t,C,DC/Dx))/Dx + s(x,t,C,DC/Dx) , where m=0, so c*dC/dt =
% df/dx + s
function [c,f,s] = mypde(x,t,C,DCDx)
c = 1;
D = 0;
% since our slab is composed of two parts with different diffusion
```

```
% constants, we need to define D depending on the distance away from te
% origin
D1 = 1e-10; %diffusion constant of stent (cm2 s-1)
D2 = 7e-8; %diffusion constant of artery wall (cm2 s-1)
L1 = 5e-4; %thickness of stent (cm)
L2 = 0.01; %thickness of artery wall (cm)
if x<L1
   D = D1;
else
   D = D2;
end
f = D*DCDx;
s = 0;
 % function to define initial conditions
function c0 = icpde(x)
% initial condition at x = 0
c0 = 0;
 % function to define boundary conditions
% p and q for left and right side boundaries are determined to fit:
% p(x,t,u) + q(x,t) * f(x,t,u,Du/Dx) = 0
function [pl,ql,pr,qr] = bcpde(xl,cl,xr,cr,t)
Cmax = 1; %maximum concentration (M)
% in the left side we have cl = Cmax so, cl - Cmax =0 fitting to pl+ql*f =0
pl = cl - Cmax;
ql = 0;
```

```
% in the right side we have dc/dx = 0 , so to fit p + q*du/dx =0

pr = 0;

qr = 1;
```

Matlab Code (Cylindrical Coordinates Method):

This code presents numerical analysis of diffusion equation in cylindrical coordinates using MATLAB PDEPE method.

```
function presentation_cylindrical()

%define constants taken from the research paper

 L1 = 5e-4; % stent thickness

 L2 = 0.01; % artery wall thickness

 Rout = 0.341; % outside vein radius

 Rin = Rout - L1 - L2; %inside vein radius

 x = linspace( Rin, Rout, 100); % distance variable starting from inside radius going to outside radius

 t = linspace(0,120,100); % time variable

 % pdepe method in cylindrical coordinates

sol = pdepe(1,@mypde,@icpde,@bcpde,x,t);

% Extract the first solution component as C.

C = sol(:,:,1);

% A surface plot representing the concentration vs. distance and time

figure(1)

surf(x,t,C);

title('PDEPE Approximation Method in cylindrical coordinates');

xlabel('Distance (cm)'), ylabel('Time (s)'), zlabel('Concentration (M)');

% A solution profile observing concentration change in certain distances

% over time
```

```
figure(2)
plot(t,C(:,12),'m')
hold on
plot(t,C(:,24),'b')
hold on
plot(t,C(:,262),'g')
hold on
plot(t,C(:,499), 'r')
title('Drug Concentration profile at particular distances');
legend('x=L1/2','x=L1','x=L1+L2/2','x=L1+L2');
xlabel('Time (s)'), ylabel('CONCENTRATION (M)');
% A solution profile observing concentration change in certain times
% over distance
figure (3)
plot (x, C(5,:),'b')
hold on
plot (x, C(50,:), 'r')
hold on
plot (x, C(499,:), 'g')
title('Drug concentration profiles in different times');
legend ('t=1.2s', 't=12s','t=120s');
xlabel ('Distance (cm)'), ylabel('Concentration (M)');
 % function to define the components of pde as Matlab can recognize
% from the general form D*d2C/dx2 = dC/dt
% find c, f, s such that our equation fits c(x,t,C,DC/Dx) * DC/Dt = x^(-m)
```

```
% * D(x^m * f(x,t,C,DC/Dx))/Dx + s(x,t,C,DC/Dx) , where m=0, so c*dC/dt =
% df/dx + s
function [c,f,s] = mypde(x,t,u,DCDx)
c = 1;
% since our slab is composed of two parts with different diffusion
% constants, we need to define D depending on the distance away from te
% origin
D = 0;
D1 = 1e-10; %diffusion constant of stent (cm2 s-1)
D2 = 7e-8; %diffusion constant of artery wall (cm2 s-1)
L1 = 5e-4; %thickness of stent (cm)
L2 = 0.01; %thickness of artery wall (cm)
if x<L1
   D = D1;
else
   D = D2;
end
f = D*DCDx;
s = 0;
 % function to define initial conditions
function c0 = icpde(x)
% initial condition at x = 0
c0 = 0;
% function to define boundry conditions
% p and q for left and right side boundries are determined to fit:
```

```
% p(x,t,u) + q(x,t) * f(x,t,u,Du/Dx) = 0

function [pl,ql,pr,qr] = bcpde(xl,cl,xr,cr,t)

Cmax = 1; %maximum concentration (M)

% in the left side we have cl = Cmax so, cl - Cmax =0 fitting to pl+ql*f =0

pl = cl - Cmax;

ql = 0;

% in the right side we have dc/dx = 0 , so to fit p + q*du/dx =0

pr = 0;

qr = 1;
```

4. CONCLUSION:

This problem has been solved in both rectangular and cylindrical coordinates. We can find the most accurate method of solving this problem by calculating the boundary layer thickness.

Boundary Layer Thickness Approximation:

$$\delta = \sqrt{4D_2 t}$$

4. Time to reach steady-state for cylindrical coordinates (found by MATLAB simulation) is 20 s.

$$\delta = \sqrt{4 * 7 * E - 8 * 20}$$

5. $$\delta = 0.0024\ cm$$

6. Actual length through which diffusion is taking place is 0.01 cm. This is almost 4-fold larger than boundary layer thickness. Therefore, solving the problem in Cartesian coordinate system is not a very accurate method. It can thus be seen that solving this problem in cylindrical coordinate system is a more accurate technique than by Cartesian system.

The numerical and analytical analysis shows that the drug gets diffused through the arterial wall in a very short period of time. Considering that the stent is coated with only the drug, and that the thickness of the arterial wall and of the stent is very small, this is a reasonable conclusion. However, in actuality,

we want the stent to show sustained release. This would therefore mean that our solution indicates that another top-coat (such as a polymer which decreases drug diffusion) must be taken into account if we are to achieve diffusion over longer periods of time, for better therapeutic efficacy.

5.FUTURE WORK:

In this problem, we have assumed the stent to be impervious to blood flow. However, if this were not the case, convection would come into play. The equation would then include terms for both diffusion and convection. Another case that can be considered is that of a polymeric coat encapsulating the drug-coated stent. This would be a more practical application, and in this case, the problem would have three layers: one for the stent, one for the polymeric coating and one for the arterial wall. The polymeric coat would decrease the rate of diffusion of the drug, thus leading to a longer duration of action.

REFERENCES:

1. Thomas Lemke, David Williams, (2006), *Foye's principles of medicinal chemistry-6th edition*, USA: Lippincott, Williams & Wilkins.
2. G. Pontrelli, F. de Monte; (2007). Mass diffusion through two-layer porous media: an application to the drug-eluting stent. *Int. J. Heat Mass Transf.*, Vol. 50, pp. 3658-3669
3. *C.* J. van Andel, P. V. Pistecky, C. Borst; (2001). Mechanical Properties Of Coronary Arteries And Internal Mammary Arteries Beyond Physiological Deformations, *Engineering in Medicine and Biology Society*, vol 1, pp. 113 - 115