BENG 221 Grant Vousden-Dishington
Problem Solving Report

The Minimum-Jerk Trajectory for n-DOF Reaching Movements via Calculus of Variations

PROBLEM STATEMENT

Efficient and accurate motor control remains an elusive solution within the field of robotics, and the success of a control system depends heavily on what metric is used to define the path taken by the motor plant. Calculating the desired movements of a robot with three or more degrees of freedom is an extremely complex problem, but the central nervous system controls dozens of degrees of freedom very quickly. As such, there is much interest in discovering the policy used by the motor nervous system and how it changes. Because fields, including bioengineering, aim to understand, implement, and even restore such functions in disabled patients, validating computational models of trajectory planning policy in simulation is paramount to that discovery.

In the motor control framework, this policy must include a description of the trajectory the end effector must take – here, “end effector” refers to the (zero-)point on the arm that we desire to reach the target destination. By default, the end effector is the hand or gripper of the system, but it may just as well be the functional point of a tool, like a hammer or wrench, or another point on the limb, such as the center of the forearm or shins. The trajectory of a system is more complex than simply naming the Cartesian coordinates along the path; a trajectory function must describe *when* the end effector is expected to be at these positions. The problem here is that, given any starting and ending position in Cartesian space, there are infinitely many trajectories. Namely, we can make the trajectory as smooth or noisy as we like, even when there are some constraints on where we can move to.

This is the motivation for formulating the trajectory-forming problem as an optimization problem. Though not necessarily unique, solutions for this type of problem will at least meet the criteria good enough for our purposes. Thus, the next part of the problem is to decide what must be optimized. In motor control, there are primarily two outputs that can be optimized: jerk and energy. Optimizing jerk entails finding the smoothest curve connecting an initial position, velocity, and acceleration to a desired position, velocity, and acceleration over an *already specified* time interval. Minimum energy models are similar, but they are more complex in that the energy expended by a plant is dependent on the architecture of the actuators/musculature, and the time-constraints for these models is more relaxed, since shorter time intervals typically involve expending more energy because a higher peak velocity is needed. Of key importance is that minimum-energy models necessarily rely on the dynamics of the system but minimum-jerk models can suffice on a simply kinematic description. It is for this reason that I use a simplified minimum-jerk model instead.

CALCULUS OF VARIATIONS THEORY AND SETUP

Calculus of variations is similar to gradient descent methods in differential equation analysis. Like gradient descent, it is a first order approach focused on minimizing the gradient of a given function ∇F . However, in the case of gradient descent, F may be any type of function, but calculus of variations deals specifically with *functionals* F, maps which take other functions as parameters and return elements that form the basis of that function's vector space, which is typically a scalar, as it will be in this formulation. The functional will serve as a 'cost' function that needs to be minimized. For this experiment, the cost function being minimized is jerk, which is defined as the third-derivative of position, the rate of change of acceleration.

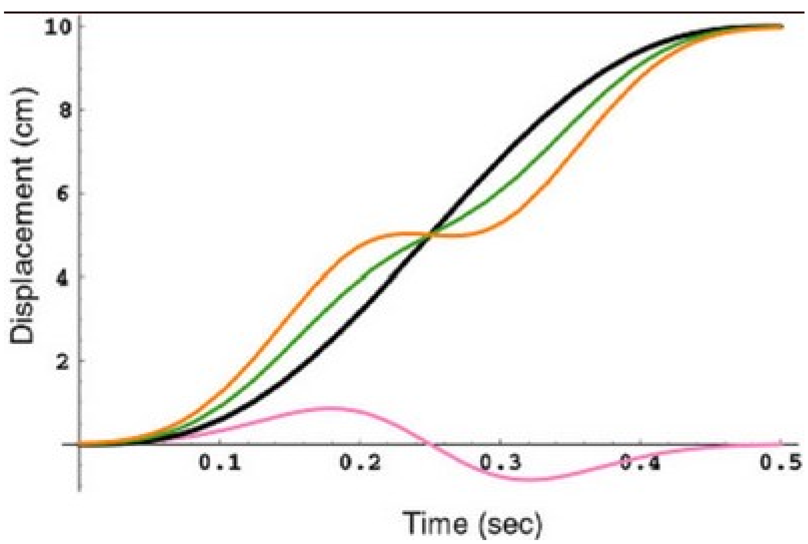
$$\dddot{x}(t)$$

We define jerk as only a function of time to simplify the calculations. In realistic conditions, other factors will determine the jerk, such as forces due the the environment and muscle weakness and fatigue. As such, the definition of jerk is easily extendable to other domains and partial differential equations. In order to turn this definition of jerk into a cost, we must ensure that it is always positive and take the totality of this measurement over the entire movement. As each movement is continuous, we define this sum as an integral over the time of the movement t_I to t_f, the initial and final time points, respectively.

$$C(t)=\int_{t_i}^{t_f}(\dddot{x}(t))^2\,dt$$

Calculus of variations works by finding not the minimum of this functional, but its minimum with respect to an arbitrary perturbation function, which we will call *u(t)*. This perturbation function is multiplied by a perturbation constant α and added to the original jerk function. Below is shown the total function's new form, as well as a graph of how it affects the overall position and trajectory.The plot, modified from Shadmehr et al. (2005), shows the original function $x(t)$ in black, the pertubation function *u(t)* in pink, and the sum of the two, with an α equal to 1 (green) and greater than 1 (orange).

$$x(t)+\alpha u(t)$$



Thus, the final form of the functional that we will use for this problem has the form

$$C(\boldsymbol{r}(t))\rightarrow C(\boldsymbol{r}(t)+\alpha u(t))$$

To be useful, we must first apply some constraints to the perturbation function. The key aspects we wish to capture, as shown in the plot above, are that the perturbation function and all of its defined derivatives *vanish* at the endpoints t_i and t_f. Because we are dealing with the third-derivative of position, we need to make sure *u(t)* and its first three derivatives are zero.

$$u(t_i)=\dot{u}(t_i)=\ddot{u}(t_i)=\dddot{u}(t_i)=0$$

$$u(t_f)=\dot{u}(t_f)=\ddot{u}(t_f)=\dddot{u}(t_f)=0$$

Just as in single-variable optimization and gradient descent, calculus of variations takes the derivative of the cost functional. Instead of just seeking where the functional itself equals 0, however, we also evaluate the integral when the perturbation constant α equals 0, and use integration by parts to find which derivative of position must equal zero for the cost (i.e. jerk) to be minimized. In addition to this mapping from function to functional, a critical step to using calculus of variations that allows us to determine which derivative of *x(t)* must equal zero is the *fundamental lemma of calculus of variations.* This lemma states the following: If it is the case that, given a k-times continuously differentiable function *f*,

$$\int_a^b f(x)h(x)\,dx=0$$

for every k-times continuously differentiable function *h(x)* such that *h(a) = h(b) = 0,* then f(x) is identically zero on the entire interval [a,b]. We can see from our definition of *u(t)* above that it satisfies the conditions for *h(x)*. This lemma will allow us to know the order of the derivative of ***r(t)*** that must equal zero in order to minimize the jerk.

SIMPLIFIED ANALYTICAL SOLUTION

The first step in our derivation is to take the derivative of the function with respect to the perturbation variable α. Then, we evaluate this derivative at the point where this variable equals 0 to characterize the point where the perturbation is minimized.

$$C(\boldsymbol{r}(t)+\alpha u(t))=\frac{1}{2}\int_{t_i}^{t_f}(\ddot{r}(t)+\alpha\ddot{u}(t))^2\,dt$$

$$\frac{\partial C(\boldsymbol{r}+\alpha u)}{\partial\alpha}=\int_{t_i}^{t_f}(\ddot{r}(t)+\alpha\ddot{u}(t))\ddot{u}(t)\,dt$$

$$\frac{\partial C(\boldsymbol{r}+\alpha u)}{\partial\alpha}\Big|_{@\alpha=0}=\int_{t_i}^{t_f}\ddot{r}(t)\ddot{u}(t)\,dt$$

Now that we have our integral, the next step is to use integration by parts to iteratively redefine this integral until we find the point where u(t) = 0. Recall that the perturbation function u(t) vanishes at the endpoints t_i and t_f.

$$\int_{t_i}^{t_f}\ddot{\boldsymbol{r}}(t)\ddot{u}(t)\,dt=\ddot{\boldsymbol{r}}(t)\ddot{u}(t)\Big|_{t_i}^{t_f}-\int_{t_i}^{t_f}\boldsymbol{r}^{(4)}(t)\ddot{u}(t)=-\int_{t_i}^{t_f}\boldsymbol{r}^{(4)}(t)\ddot{u}(t)$$

$$-\int_{t_i}^{t_f}\boldsymbol{r}^{(4)}(t)\ddot{u}(t)\,dt=-\boldsymbol{r}^{(4)}(t)\dot{u}(t)\Big|_{t_i}^{t_f}+\int_{t_i}^{t_f}\boldsymbol{r}^{(5)}(t)\dot{u}(t)=\int_{t_i}^{t_f}\boldsymbol{r}^{(5)}(t)\dot{u}(t)$$

$$\int_{t_i}^{t_f}\boldsymbol{r}^{(5)}(t)\dot{u}(t)\,dt=\boldsymbol{r}^{(5)}(t)\dot{u}(t)\Big|_{t_i}^{t_f}-\int_{t_i}^{t_f}\boldsymbol{r}^{(6)}(t)u(t)=-\int_{t_i}^{t_f}\boldsymbol{r}^{(6)}(t)u(t)=0$$

The final equality satisfies the fundamental lemma of calculus of variations. By that lemma, we can state that $\mathbf{r}^{(6)}=0$. Thus, we can conclude that *any position function whose sixth derivative is zero minimizes the jerk.* To make the sixth derivative of a function zero, we guess that the equation for position has the general form of a fifth-order polynomial.

$$\boldsymbol{r}(t)=A+Bt+Dt^2+Et^3+Ft^4+Ht^5$$

$$\dot{\boldsymbol{r}}(t)=B+2\mathrm{Dt}+3\mathrm{Et}^2+4\mathrm{Ft}^3+5\mathrm{Ht}^4$$

$$\ddot{\boldsymbol{r}}(t)=2\mathrm{D}+6\mathrm{Et}+12\mathrm{Ft}^2+20\mathrm{Ht}^3$$

The values of the coefficients are, as usual, found by applying the known properties of the problem, defined in the previous section. More generally, the trajectory depends only on the duration of the movement, rather than the absolute starting and ending times, t_i and t_f. Since it is only the difference we are interested in, we can reset t_i=0 and $t_f = t_f - t_i$. Assuming that the end-effector begins and ends the movement at rest (i.e. no acceleration or velocity initially), we find that A, B, and D equal x_i, 0, and 0, respectively. The values of E, F, and H depend on what the value of t_f is. Namely, the following three equations must be met in order to solve for E, F, and H.

$$\boldsymbol{r}(t_f)=x_f \Rightarrow Et_f^3+Ft_f^4+Ht_f^5$$
$$\dot{\boldsymbol{r}}(t_f)=0 \Rightarrow 3\,Et_f^3+4\,Ft_f^4+5\,Ht_f^5$$
$$\ddot{\boldsymbol{r}}(t_f)=0 \Rightarrow 6\,Et_f^3+12\,Ft_f^4+20\,Ht_f^5$$

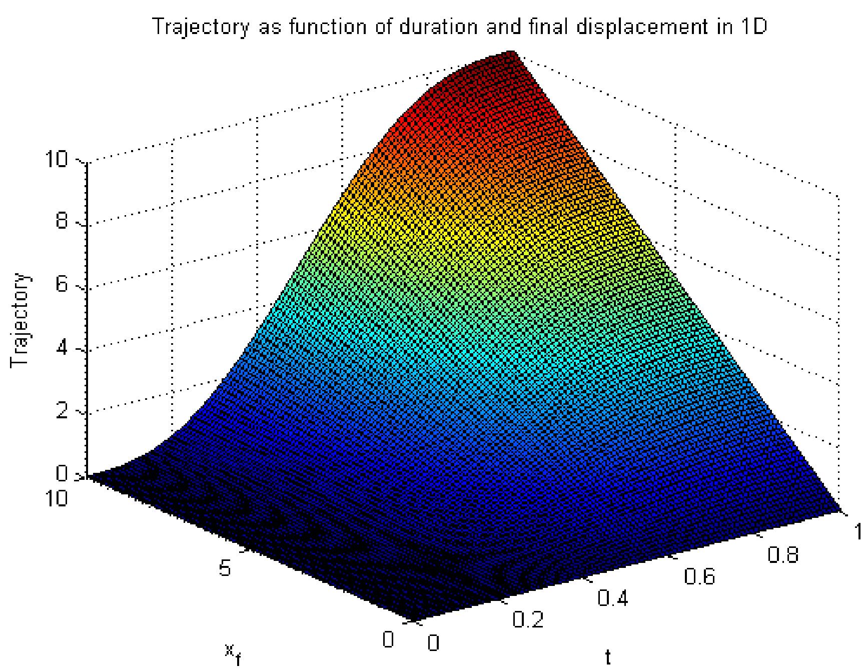
Symbolically manipulating the equation into matrix form gives an easy method for determining the values of E, F, and H. As noted by Flash & Hogan (1985), the general form of this equation comes out to a relatively simple expression.

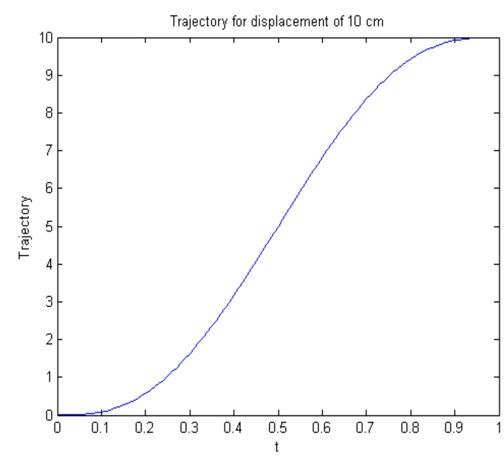
$$\boldsymbol{r}(t)=\begin{bmatrix} x_{1\mathrm{i}}+(x_{1\mathrm{f}}-x_{1\mathrm{i}})(10(t/d)^3-15(t/d)^4+6(t/d)^5) \\ x_{2\mathrm{i}}+(x_{2\mathrm{f}}-x_{2\mathrm{i}})(10(t/d)^3-15(t/d)^4+6(t/d)^5) \\ \vdots \\ x_{ni}+(x_{nf}-x_{ni})(10(t/d)^3-15(t/d)^4+6(t/d)^5) \end{bmatrix}$$

In this trajectory, each element in the vector corresponds to one of the degrees of freedom. It's easy to see that each element follows the same pattern and dependency on the time and duration. Another, less obvious but equally important consequence of the above form is that any minimum-jerk trajectory in two or three dimensions is necessarily a straight line.

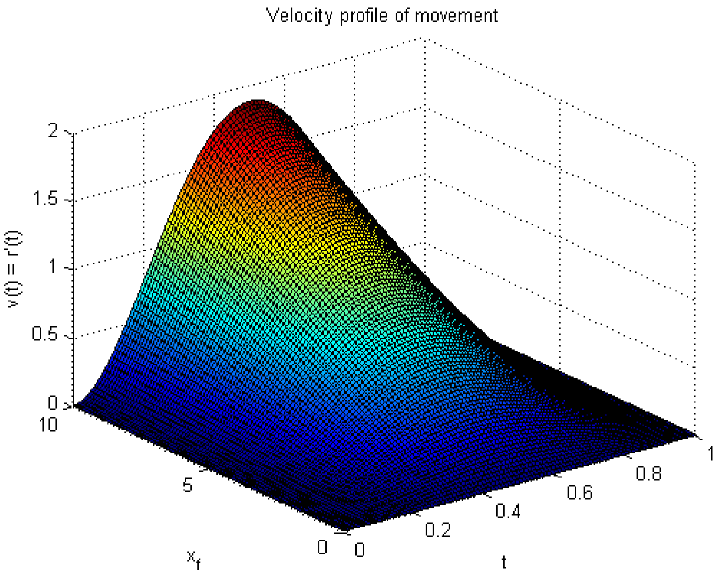
NUMERICAL VALIDATION

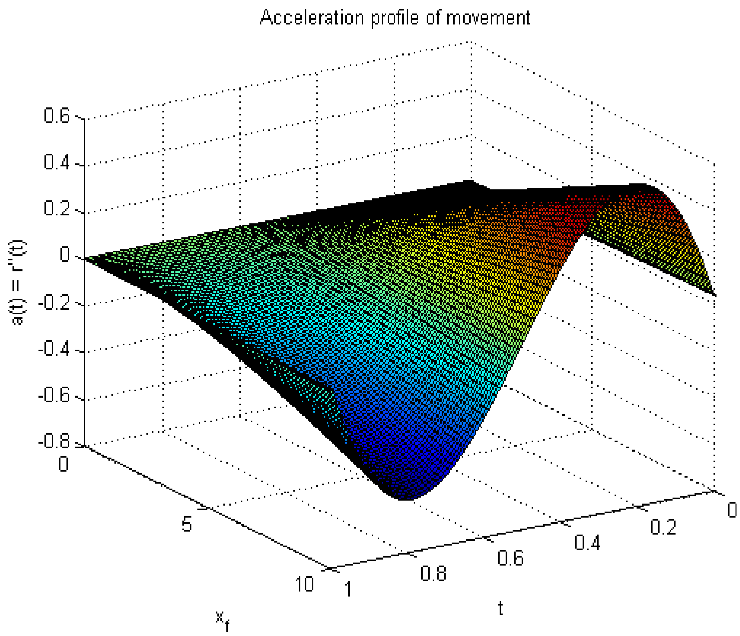
The previous section mentioned the importance of the overall duration time to the overall minimum-jerk trajectory. Biologically reasonable times for reaching movements in humans will typically be between 0.5 and 1 second. Below is displayed a surface plot demonstrating the change in trajectory (i.e. expected position with respect to time) against different time-points and total displacements (given as the magnitude) for the analytical solution.



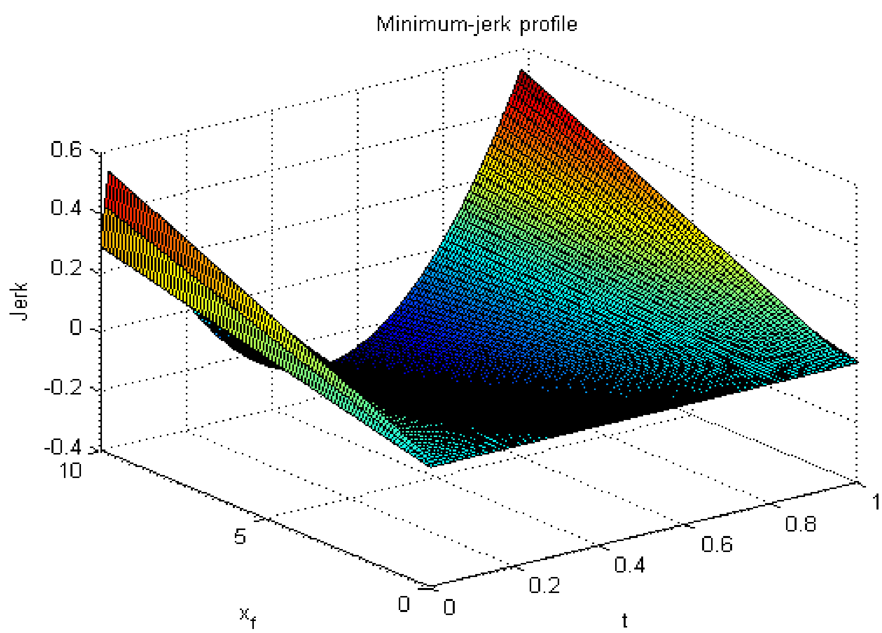


The primary effect of increasing the displacement is, naturally, an increase in the gain of the sigmoid shape of the plot. The figure to the right shows the cross section of the graph with respect to time, more clearly showing the sigmoid shape of the overal trajectory. Similarly, we show the change in the velocity and acceleration profiles over differing displacements, calculated numerically based off the given trajectory points. As expected, there is bell-shaped velocity profile, indicating the limb reaches maximum velocity at the mid-point of the total movement. This shape is negligibly different from human psychophysical experiments, seen in the text by Shadmehr et al. Note that the displacement-axis on the acceleration profile is reversed, to more clearly show the sinusoidal shape.





Finally, we show the minimum-jerk profile, which is the aspect we aim to minimize. One key aspect to note is that the jerk is inversely proportional to velocity. That is, the faster we make a movement, the more we minimize jerk. If we expect the motor system to plan trajectories based on minimum jerk, we also infer that it plans trajectories with the highest peak-velocity.



DISCUSSION

Optimization of the movement trajectory for an end-effector is a difficult problem in motor control research. What is presented here is the most simplified version of the problem that demonstrates useful results. I considered a the general case of an n-dimensional end-effector that in a reaching motion, providing there are no obstacles that might impede the movement of the end-effector and that the task is discrete, not part of a more complex set of goals. I further simplified our solution to be variable with time and constant or symmetric about any other variables. In this case, I reproduced the bell-shaped velocity profile and the sigmoid position profile observed in psychophysical experiments. One of the most important numerical results shown is the inverse proportionality between velocity and jerk. The symmetry of these curves will be broken if we assume different starting and ending conditions, namely, that the initial velocity or acceleration may be different from the final values, which would be the case if this reaching movement were only one step of a more complex plan, for example, in reaching for an object with the intent of moving it to another location.

Another issue to discuss, as is often mentioned in discussing optimal motor control policies, is whether or not there is another function which should be optimized. That is, could jerk be defined in terms of another derivative of position, such as fourth, fifth, or sixth? Perhaps, optimizing acceleration, the second derivative of position, would work just as well. The best policy should be that which most accurately matches the observations of psychophysical experiments in humans and primates. A common way of measuring this accuracy is to measure the ratio of peak-velocity to average-velocity. In humans, this ratio is observed to be about 1.75, so the average velocity is two-thirds of the global maximum during the movement. In the velocity data gathered above, this ration was 2.0, for every value of displacement – The value cited measured by Shadmehr et al. (2005) is 1.825. The difference may be due to resolution along time. Experiments taking the second and fourth derivatives as the function to be optimized resulted in values of 1.5 and 2.186, respectively. Thus, the higher the order of the derivative we use to optimize our function, the greater the ratio becomes. Thus, the psychophysical value of the ratio falls exactly half-way between the measured value optimizing the third derivative and the cited value optimized the second derivative - though the cited value for the third derivative is closer than the one I measured. As such, we may hypothesize that the 'function' optimized by the nervous system is a fractal derivative in between the second and third. Therefore, optimizing jerk provides one of the optimal approximations of motor planning computation.

REFERENCES

Reza Shadmehr and Stephen P. Wise. *The computational neurobiology of reaching and pointing: a foundation for motor learning.* MIT Press, 2005.