

Modeling Ultrasonic Wave Propagation Through the Kidney

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Table of Contents

Introduction	3
Model.....	3
Problem Statement:.....	3
Assumptions.....	4
Analytical solution.....	4
Numerical Solution	8
Conclusions	11
Future Works	11
References	13
Appendix: Matlab code.....	14

Introduction

Ultrasound is any sound above the threshold of human hearing (about 20 kHz). For medical applications, the range for ultrasound is usually between 2 MHz to 20MHz [1]. These waves are mechanical waves, which are different from electromagnetic waves. This means that the pressure sound waves require a medium to travel through because the pressure wave travels by displacing particles in the air [2].

Imaging is one of the main applications of ultrasound technology in medicine. It functions similar to radar technology, where the waves are emitted and reflected back to a receiver. The receiver can then form an image of the object based on the reflected waves. There are many other medical applications of ultrasound other than diagnostic imaging. Potential applications include: tissue ablation (cancer), breaking blood clots (sonothrombolysis), breaking kidney stones (lithotripsy), and promoting healing of muscle and other tissue [1]. This project focuses on lithotripsy, the breaking of kidney stones with pulsed ultrasonic waves.

One of the main functions of the kidneys is to remove metabolic products and toxins from blood and excreting these toxins via urine [3]. A kidney stone is a solid material that forms in the kidney when there are high concentrations of certain substances present in the urine [4]. Most of these kidney stones (75%) are comprised of calcium oxalate [5]. The stones may also contain Hydroxyapatite, uric acid or stuvite [5]. During stone formation, the concentration of calcium oxalate is usually 7 to 11 times its solubility causing crystals to nucleate [5]. These stones can vary in size, where smaller stones may pass through the urinary tract with relatively little pain. Larger stones can block the ureter and cause symptoms ranging from blood in the urine to a sharp pain in the back or lower abdomen area [4].

Current treatments for large kidney stones are uteroscopy, percutaneous nephrolithotomy, and lithotripsy [4]. Out of the three options, lithotripsy is the only non-invasive method. As hinted earlier, lithotripsy utilizes ultrasound waves to fragment the stones into smaller pieces that can exit more easily. The lithotripters create two forces on the stone: a direct force from the shock wave and a force due to the rapid expansion and collapse of air bubbles [2]. These forces help to break apart the stones, but they may also cause collateral damage to the surrounding tissues [2]. Little is known about the exact process by which the stones are fractured and the potential effects of the pressure waves on the body.

Model

Problem Statement:

In order to examine the propagation of the sound wave inside of the kidney we will consider a transducer placed directly against the kidney. The stone is then located on the other side of the kidney and a sinusoidal input wave is produced by the transducer at the boundary of the kidney. We will begin with a model of the kidney and stone using the 1D wave equation to track the longitudinal waves produced by the ultrasound pressure.

Assumptions

In this report, we modeled the propagation of ultrasound in the kidney tissue. For this model, the assumptions we made were:

1. The model has two boundaries. One is the surface of kidney, where we put the ultrasound transducer. The other side is the surface of the kidney stone.
2. All the kidney tissue in the middle is homogeneous and the ultrasonic wave would not be absorbed by it.
3. The surface of the kidney stone is incompressible and all the waves will be reflected back when they hit the surface.
4. There is a transducer placed on the surface of kidney as a source, generating a scaled sinusoidal wave which is the function of time.

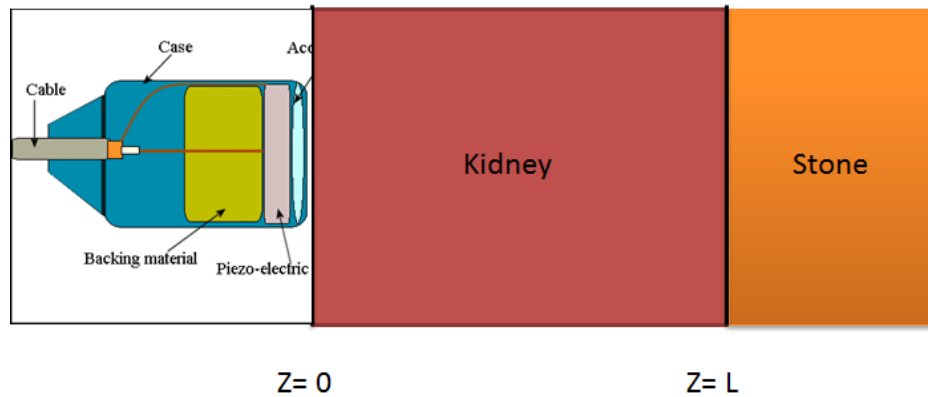


Figure 1: Simplified 1D model of ultrasound hitting an incompressible kidney stone

Analytical solution

For the simplest approach, we can focus on the one dimension situation. In such case, wave equation and its boundary conditions are expressed below:

Wave equation:

$$\frac{\partial^2 P(z,t)}{\partial t^2} = c^2 \cdot \frac{\partial^2 P(z,t)}{\partial z^2}$$

Boundary conditions:

$$P(0,t) = A_0 \cos \omega_0 t$$

$$\frac{\partial}{\partial z} P(L,t) = 0$$

In this case, we have a value-flux type boundary condition. The transducer on one side will keep generating a cosine wave with angular frequency ω_0 and amplitude A_0 . On the other side, the flux is zero because the surface of the kidney stone is incompressible and reflects all waves without changing their phases and amplitude.

Because the time-varying component is harmonic oscillator, we can have a solution form like below:

$$P(z,t) = \phi(z) \cdot e^{j\omega t}$$

After applying the separation of variables to the 1D wave equation we get:

$$-\omega^2 \cdot e^{j\omega t} \cdot \phi(z) = c^2 \cdot e^{j\omega t} \cdot \frac{d^2\phi}{dz^2}$$

$$\frac{d^2\phi}{dz^2} + \frac{\omega^2}{c^2} \phi(z) = 0$$

Coefficients we are using in this case:

$$k^2 = \frac{\omega^2}{c^2}$$

$$k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$$

is the angular frequency of the wave propagation, k is wave number here, λ is the wave length.

$$\frac{d^2\phi}{dz^2} + k^2 \cdot \phi(z) = 0$$

Using characteristic equation to solve this problem:

$$r^2 + k^2 = 0$$

So the eigenvalues are

$$r_{1,2} = \pm j \cdot k$$

Thus the solution to (1) is:

$$\phi(z) = A \cdot e^{jkz} + B \cdot e^{-jkz}$$

Then we have our general solution to the one dimension wave equation:

$$P(z,t) = \phi(z) \cdot e^{j\omega t} = (A \cdot e^{jkz} + B \cdot e^{-jkz}) \cdot e^{j\omega t}$$

To solve A and B, we apply BCs :

$$P(0,t) = (A + B) \cdot e^{j\omega t} = \cos \omega_0 t$$

$$\frac{\partial}{\partial z} P(L,t) = -jk \cdot e^{j\omega t} (A \cdot e^{jkL} + B \cdot e^{-jkL}) = 0$$

Since $-jk \cdot e^{j\omega t}$ cannot be zero, then

$$A \cdot e^{jkL} + B \cdot e^{-jkL} = 0 \rightarrow A = B \cdot e^{-j2kL}$$

$$(A + B) \cdot e^{j\omega t} = \cos \omega_0 t \rightarrow A + B = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2e^{j\omega t}} = \frac{e^{j(\omega_0 - \omega)t} + e^{-j(\omega_0 + \omega)t}}{2}$$

Finally we get:

$$A = \frac{e^{j(\omega_0 - \omega)t} + e^{-j(\omega_0 + \omega)t}}{2(1 + e^{j2kL})}$$

$$B = \frac{e^{j(\omega_0 - \omega)t} + e^{-j(\omega_0 + \omega)t}}{2(1 + e^{-j2kL})}$$

A and B are both the functions of time.

The analytical solution is:

$$P(z,t) = \left(\frac{e^{j(\omega_0 - \omega)t} + e^{-j(\omega_0 + \omega)t}}{2(1 + e^{j2kL})} \cdot e^{jkz} + \frac{e^{j(\omega_0 - \omega)t} + e^{-j(\omega_0 + \omega)t}}{2(1 + e^{-j2kL})} \cdot e^{-jkz} \right) \cdot e^{j\omega t}$$

After simplifying , the imaginary parts are eliminated resulting in the following:

$$P(z,t) = \left(\frac{\cos(kz) + \cos(k \cdot (z - 2L))}{1 + \cos(2kL)} \right) \cdot \cos(\omega_0 t)$$

High Frequency Sound Wave

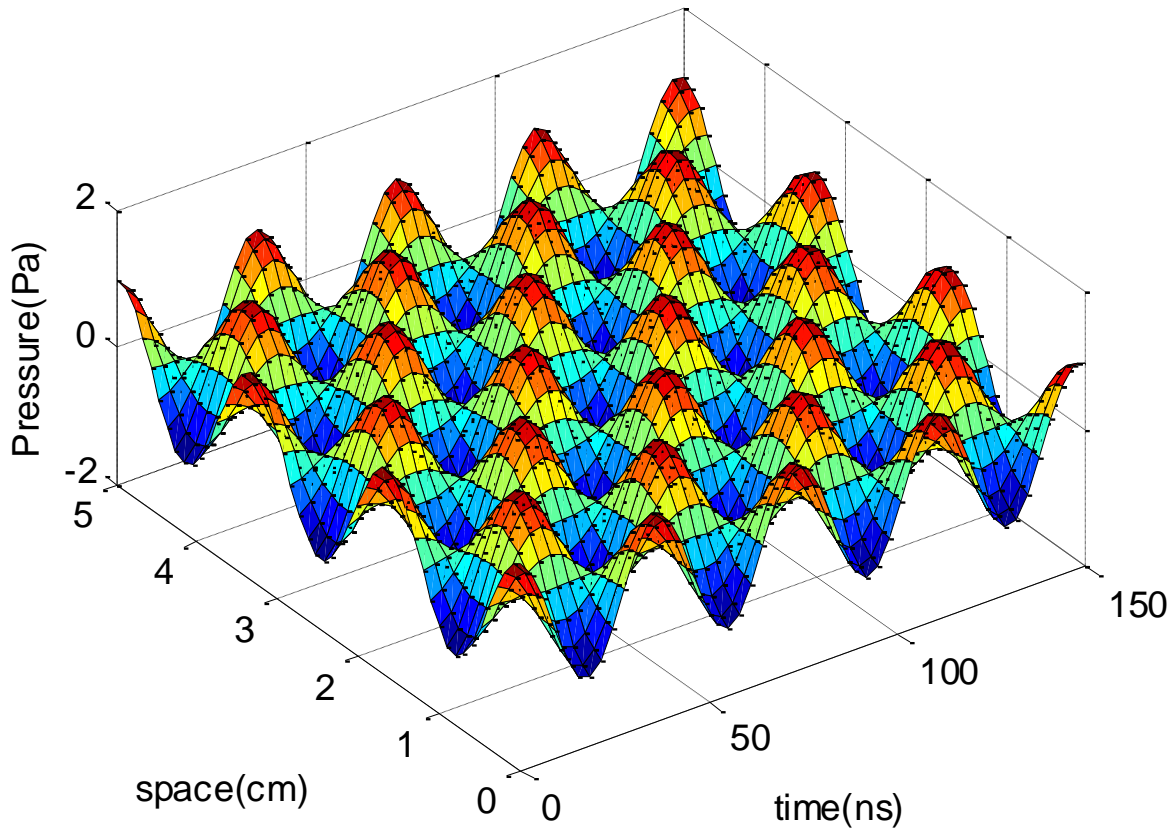


Figure 2: plot of our analytical solution in time and space

Figure 2 shows a plot of the analytical solution. It shows the cosine wave over time on the time axis. This follows our boundary condition of $\cos(\omega_0 t)$. There are cosine waves in the space direction, which is what the analytical solution predicted (combination of cosine waves). Therefore, since the time and space axes are both cosine waves, a matrix of cosine waves is created. Basically, the transducer sends a different starting pressure at each time point. This means that a cosine wave will propagate in the kidney slightly out of phase until it eventually becomes in phase again. These results are as expected, but they are not very interesting, since the entire wave is reflected back at the end boundary. This means that there is no amplitude attenuation or phase change, which shows in the plot. If there was a phase change, the back propagating wave would interfere with the forward propagating wave and result in a wave with a lower amplitude (possibly 0 if anti-phase). The pressure amplitude will scale with the input transducer wave. If we input a wave with higher amplitude, the amplitude of the solution will scale proportionally.

Numerical Solution

For our numerical solution, we utilized the PDE toolbox, which calculates finite elements approximations to solve PDEs in 2D defined in a bounded plane. After defining the boundaries on a 2D plane, the space is broken up into smaller geometric domains, which in this case is a triangular mesh, which is assumed to be continuous.

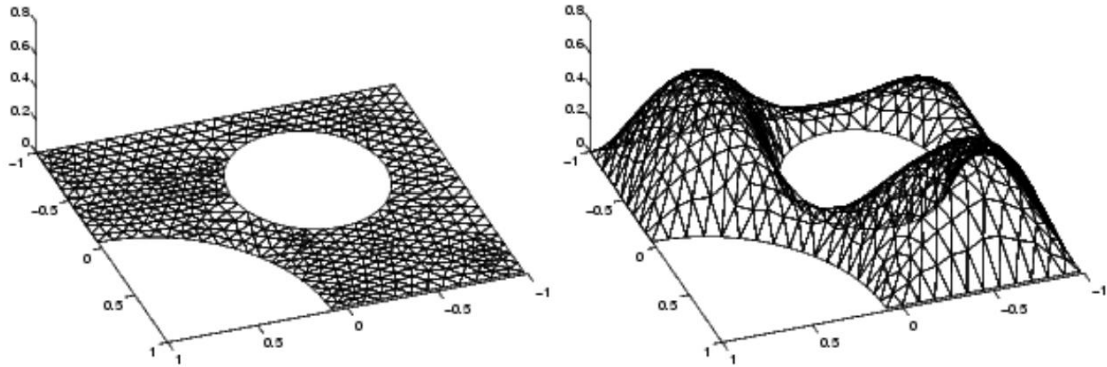


Figure 3: A Triangular Mesh (left) and a Continuous Piecewise Linear Function on That Mesh

After breaking the boundaries into meshes, the solution for each bound is calculated based on the PDE, which in our case is hyperbolic. The solution is then plotted in respect to time.

2D to 1D solution Comparison:

We wanted to compare our results for a 20MHz from our 1D analytical solution with the 2D numerical solution to see if the behavior was similar.

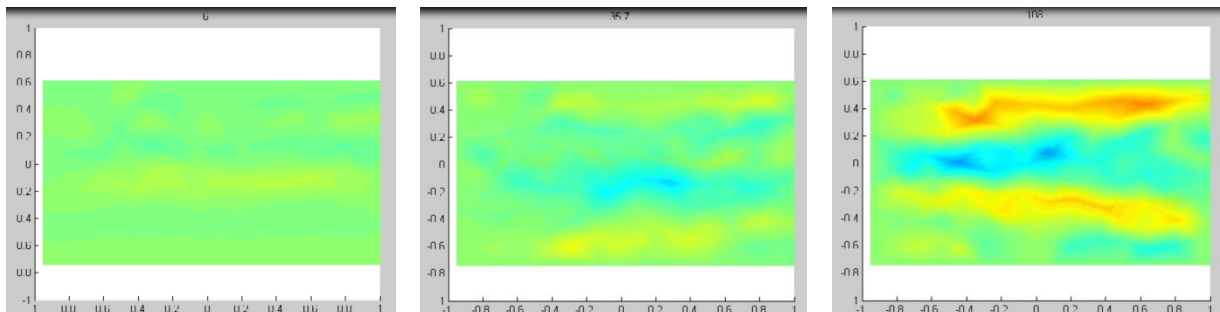


Figure 4: 2D wave model, 20 MHz cos wave generated at the top boundary and zero flux through the bottom boundary to model kidney stone layer. Time is in milliseconds.

Our results showed that after a long time, the individual waves are reflected and form larger waves with the newly generated input waves. This does align with our 1D solution which shows uniform waves initially at .15 milliseconds.

Huygen's Principle:

One way to simplify the problem of ultrasonic waves interacting with ultrasonic sound pressure waves is to assume the stones act under the Huygen's principle, which assumes that when a wave hits an object, the object becomes a point source generating another wave. The amplitude of the wave is governed by this equation:

$$U(r_0) = \frac{(U_0 e^{ikr_0})}{r_0}$$

This states that the wave from the point source will decay over time. After using the PDE toolbox in Matlab, we were able to model the one and three kidney stones acting as a source points.

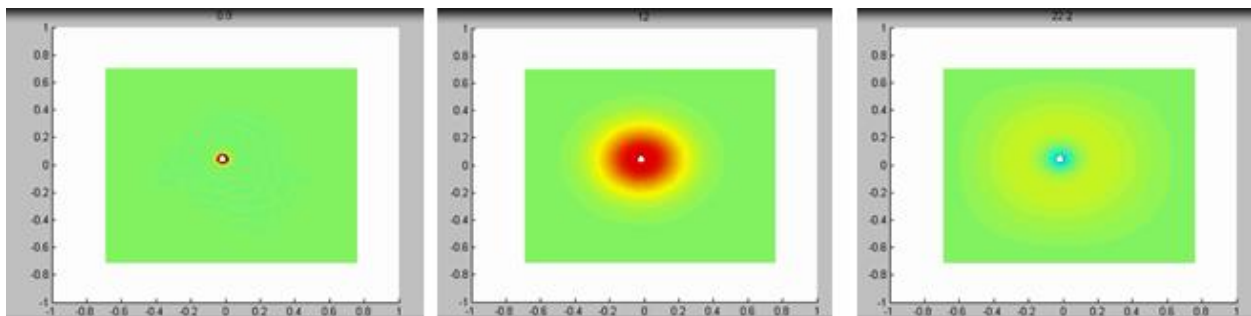


Figure 5: Kidney stone modeled using Huygen's principle. X and Y axis is position in the kidney, color is the sound pressure intensity and each plot represents a different time point in milliseconds.

The point was modeled by setting the boundary conditions to emit a wave with the previously described equation and setting an arbitrary radius of 10 centimeters from the source. Of course, this is still an approximation because we are assuming that the stone is a point source when in reality it has a finite size and shape. The stone will have

2D Solution with continuous input wave:

Although it is simpler to model the kidney stones as point sources, in reality this is rarely ever an actual occurrence. As stated earlier the transducer is placed on the surface of the skin and the waves are allowed to travel through the body. We then attempted to model a continuous planar cosine waves interacting with kidney stones that are assumed to be incompressible and thereby all waves are reflected. To do so we assumed a square slice of kidney tissue and set the input boundary to $\cos(\omega t)$ with the other three sides to have a flux of zero meaning all the waves are reflected. In addition we placed circular kidney stones in the slice and modeled them by setting the flux of the circular stones to zero. We also set other constants such as the wave velocity through the tissue to 1540 m/s [6] and the frequency.

For a wave with non-ultrasound frequency of 20 Hz this was the result:

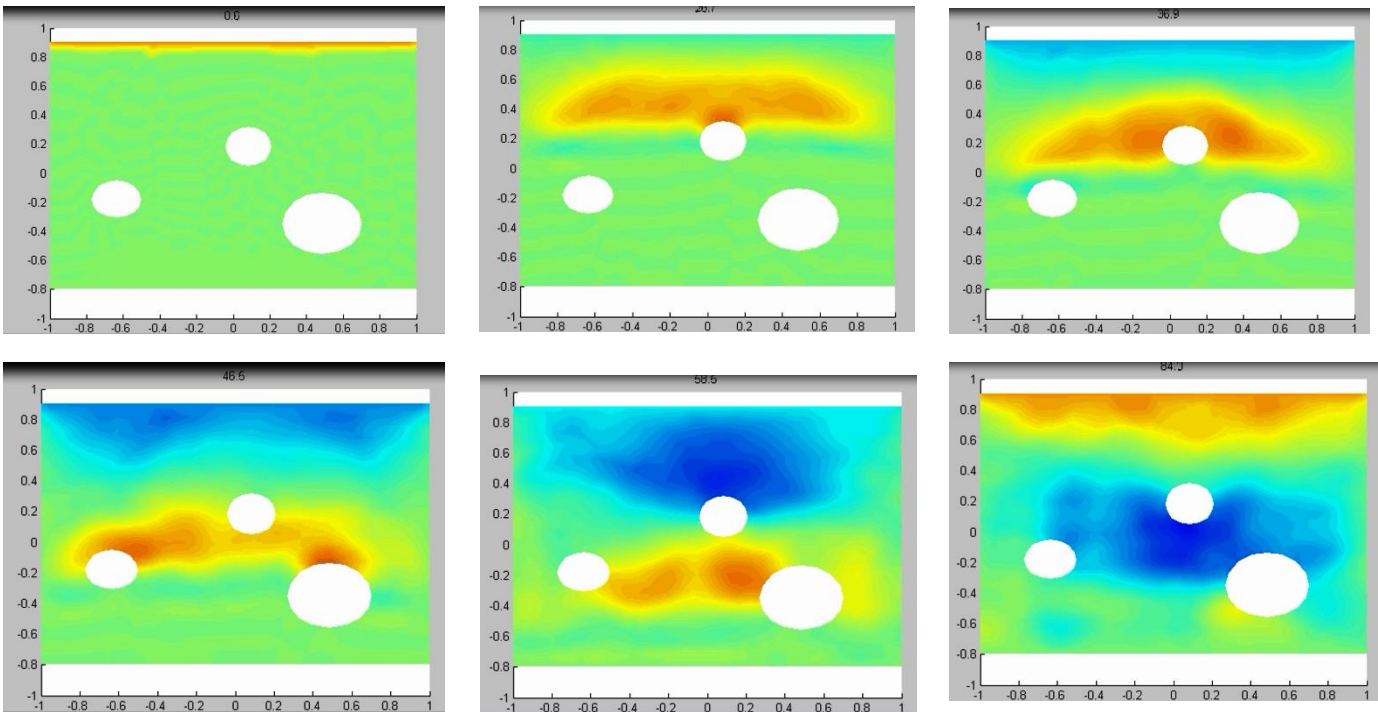


Figure 6: 2D wave model, 20 Hz wave interacting with 3 kidney stones modeled with a boundary flux of zero. Time is in milliseconds.

For the 20 Hz input wave the wave interaction between the incompressible kidney stone can be clearly distinguished and the interaction between the reflected waves can be seen as the wave is not able to pass through the stone. Moreover the constructive and destructive interactions can be observed. For an ultrasound frequency of 20 MHz this was the result:

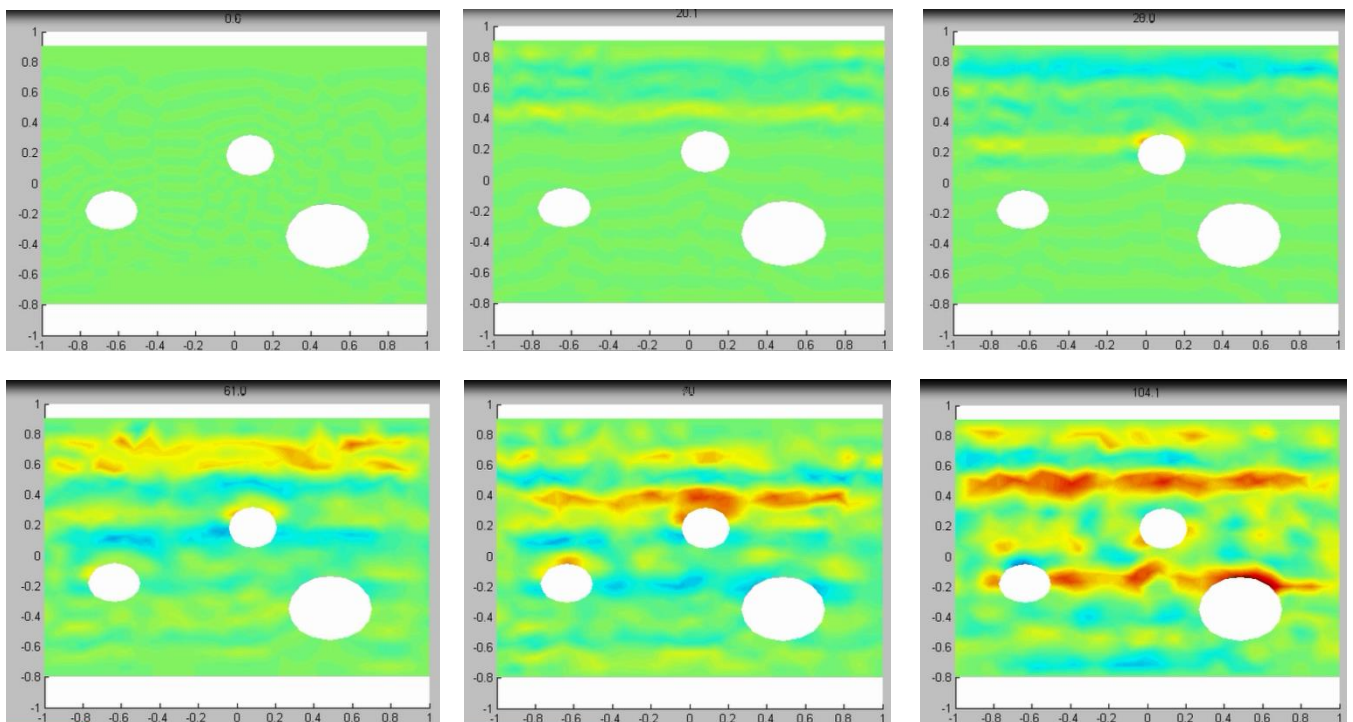


Figure 6: 2D wave model, 20 MHz wave interacting with 3 kidney stones modeled with a boundary flux of zero. Time is in milliseconds.

Since the frequency is higher, the period of oscillations should be shorter which appears in the model. One interesting behavior is the eventual constructive waves that are formed, which might be explained as a small phase shift in a period could form a wave with larger amplitude. Overall, our model appears to show that ultrasound frequencies do apply a large amplitude sound wave onto kidney stones as the treatment continues in the body.

Conclusions

Our analytical model served to demonstrate a wave propagating through a homogenous 1 dimensional tissue in response to a sinusoidal input. This gave a better understanding of the complex interactions that may occur within the kidney by first understanding the basic propagation of the wave in 1d space and time. We further complicated our system in the numerical model to closer approach the physiological situation. The system was still only a 2D system, but it showed how the wave became distorted when it encountered the zero flux holes, which represented the stones. With a time lapse video, we were able to see the pressure changes in the system. The reflected waves interacted with each other, causing both destructive and constructive interference that were capable of producing large pressures.

Future Works

This model was an extremely simplified version of the actual problem. In reality, the problem takes place in a 3 dimensional system, where the reflected waves can move with several degrees of freedom. In the future, we would like to apply the wave equation using 3d spherical coordinates. In addition, kidney stones are not circular discs or even perfect spheres. They come in a variety of shapes ranging from smooth spherical or elliptical to jagged polygons. Most likely a finite element or finite difference approach will be needed in order to accurately account for these non-symmetric shapes.

In the analytical and numerical analyses, we used homogeneous boundary conditions for the stone (e.g. zero flux). A more realistic boundary condition would be a mix of value and flux. For example a boundary condition such as $P(0, t) + \frac{\partial P}{\partial x}(0, t) = 0$. This boundary means that there will be some wave reflected and some wave absorbed, which is what would happen to the stone and surrounding tissue since there is no such material that is perfectly incompressible. In the future, for our other boundary condition, instead of a sinusoidal wave, we could use a pulsed pressure wave. Generally, most lithotripsy transducers generate a large pulse of pressure and focus it into the kidney.

Last of all, we would like to expand our model to examine the forces and stresses applied to both the kidney and stone. This has the largest practical implications as it relates directly with patient safety and the efficiency of the shockwaves in splitting the kidney stones. Once the exact mechanism for the stone fragmentation is known, the process can be streamlined to focus the waves to produced only the desired effect and minimize collateral damage to the tissue.

References

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Appendix: Matlab code

```
clear
close all
clc

c = 1560; %m/s
rhoK = 1040; %kg/m^3
rhoS = 1;
K1 = 1/(rhoK*c^2);
K2 = 1;
L = 0.05;%m

f1 = 2e6; %hz
f2 = 2e6; %hz
w = 2*pi*f1;
w0 = 2*pi*f2;
k = w*(K1*rhoK)^0.5;
n = 60;

t = linspace(0,3/f1,n); %s
z = linspace(0,L,n); %m
t1 = t*1e9; %ns
z1 = z *100; %cm

A = (exp(1i*(w0-w)*t)+exp(-1i*(w0+w)*t))/(2*(1+exp(2i*k*L)));
B = (exp(1i*(w0-w)*t)+exp(-1i*(w0+w)*t))/(2*(1+exp(-2i*k*L)));

P = ((A.*exp(1i*w*t))'*exp(1i*k*z))+((B.*exp(1i*w*t))'*exp(-1i*k*z));
% P1 = (exp(1i*w*t))'*((A.*exp(1i*k*z))+B.*exp(-1i*k*z));
figure()
surf(t1,z1,real(P));
title('High Frequency Sound Wave')
ylabel('space (cm)')
xlabel('time (ns)')
zlabel('Pressure (Pa)')
```