ON THE OCCURENCE OF STABLE HETEROCLINIC CHANNELS IN RANDOM NETWORKS

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Abstract. The Lotka-Volterra equations can be used to model the behavior of complex systems in nature. Trajectories in a stable heteroclinic channel describe transient dynamics according to the winnerless competition principle in such a system. The existence of such a channel is guaranteed if the parameters of the Lotka-Volterra equations satisfy a number of conditions. We study under what conditions a heteroclinic channel arises in a grossly organized system, i.e. a system where the coupling strength are chosen randomly. From these results we can derive what the overall structure of the system has to be.

1. Introduction

Generalized Lotka-Volterra (LV) equations can be used to model a variety of complex processes in nature. From the competition amongst different species to the modeling of firing rates in a network of inhibitory coupled neurons. A phenomenon usually observed is winnerless competition where the dynamics are governed by transient states in which one of the competitors is dominating for a period of time followed by the next temporary winner and so on. In comparison to the traditional attractor based dynamics where time goes to infinity, the transient solution is what governs the dynamics. The concept of a stable heteroclinic channel was introduced to describe such behavior in a system modeled by Lotka-Volterra equations.

The system we investigate is described by the following system of coupled ordinary differential equations given by

$$\frac{\mathrm{d}}{\mathrm{d}t}a_i(t) = \dot{a_i}(t) = a_i(t)\left(\sigma_i - \left(a_i(t) + \sum_{j\neq i}\varrho_{ij}a_j(t)\right)\right) \tag{1}$$

with $a_i(t) > 0, \sigma_i > 0$, and $\varrho_{ij} > 0$ for $i, j = 1, \ldots, N$. Alternatively with $a = (a_i)$, $\varrho_{ii} = 1$, $P = (\varrho_{ij})$ and $\sigma = (\sigma_i)$ we can state (1) in matrix form

$$\dot{a} = a * (\sigma - Pa)$$

where $*$ denotes componentwise multiplication. We call σ the *stimulus* and P the *coupling matrix*. These equations are generalized n-dimensional Lotka-Volterra equations and the original LV equations are obtained for $\sigma_i = 1$ for all i. In an application the dynamics are noisy so one would add a noise term to the last factor on the right hand side. We will omit the noise for our analytical studies.

Let Q be an saddle point in some dynamical system $\dot{x} = F(x), x \in \mathbb{R}^n$. And let $\operatorname{Re}\lambda_1 > \cdots > \operatorname{Re}\lambda_{r-1} > 0 > \operatorname{Re}\lambda_r > \cdots > \operatorname{Re}\lambda_n$ be the ordered real parts of the eigenvalues of the linearization of F at Q. Then the *saddle value* of Q

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is defined as $\nu(Q) = -\operatorname{Re}\lambda_r / \operatorname{Re}\lambda_1$, cf. [SSTC98]. Note that the points $A_i = (0,\dots,0,\sigma_i,0,\dots,0)$ with σ_i being the i-th entry are nontrivial fixed points of (1). Let $K \in \mathbb{N}, K \leq N$. The ordered tuple $\iota = (i_1,\dots,i_K)$ of pairwise different natural numbers with $1 \leq i_k \leq N$ for all k is called a *sequence of equilibria* in (1). However we allow $i_K = i_1$ and in this case the sequence of equilibria is referred to as being *closed* in contrast to the sequence being *open*. Let $I = I(\iota) = \{i_k = \pi_k(\iota) \mid \pi_k \text{ is projection on } k\text{-th coordinate}\}$.

Definition 1. *Let A_j be as above. Let ι be a sequence of equilibria and suppose A_i is a saddle point with an one dimensional unstable manifold for all $i \in I$ and that there are heteroclinic orbits $\Gamma_{i_{k-1}i_k}$ on the unstable manifold of $A_{i_{k-1}}$ that converge to A_{i_k} for all $k = 2,\dots,K$ where $K = \operatorname{card} I$. Define*

$$\Gamma(\iota) := \bigcup_{i\in I} A_i \cup \bigcup_{2\leq k\leq m} \Gamma_{i_{k-1}i_k}.$$

Then

(1) if ι is a closed sequence of equilibria then Γ is called a heteroclinic and
(2) a heteroclinic sequence otherwise.

The number $K < N$ is called the length of the heteroclinic sequence/contour. If the saddle values $\nu(A_i)$ satisfy $\nu(A_i) > 1$ for all $i \in I$ then the heteroclinic sequence is called stable.

In other words a heteroclinic contour is a closed heteroclinic sequence. A stable heteroclinic channel now is characterized by the behavior of trajectories in a vicinity of the heteroclinic sequence, that is the unstable manifolds of the fixed points have to be one dimensional and a trajectory has to stay in that neighborhood for some time $T > 0$.

Definition 2. *Assume for the sequence of equilibria ι there is a corresponding stable heteroclinic sequence $\Gamma(\iota)$. Let $\varepsilon > 0$ and let $B_k = B(A_{i_k}, \varepsilon)$ be an open ball of radius ε centered at A_{i_k}. Furthermore let $V(\iota,\varepsilon)$ be an ε-neighborhood of Γ, i.e. $V(\iota,\varepsilon) = \left\{x \in \mathbb{R}^N \mid \operatorname{dist}(x,\Gamma) < \varepsilon\right\}$ where* dist *is the euclidean distance in $\mathbb{R}^n$. We say that* (1) *has a heteroclinic channel in $V = V(\iota,\varepsilon)$ if there is a neighborhood U of A_{i_1} such that for every $a_0 \in U$ there is a $T > 0$ such that the solution $a(t,a_0)$ of* (1) *satisfies the following conditions:*

(1) $a(0,a_0) = a_0$,
(2) $a(t,a_0) \in V$ for all $t < T$,
(3) and for every $1 \leq i_k \leq K$ there exists $t_k < T$ such that $a(t_k,a_0) \in B_k$.

The definition of a heteroclinic channel given here for the LV equations coincides with the more general definition that can be found in [RHVA08]. The following conditions for the existence of a stable heteroclinic channel were derived in [AZR04].

Theorem 3 (Afraimovich, Zhigulin, Rabinovich (2004))**.** *Let ι be a sequence of equilibria. Suppose that in the context of the generalized Lotka-Volterra equations* (1)*, the following conditions are satisfied for $k = 1,\dots,K$*

$$0 < \sigma_{i_{k+1}} - \varrho_{i_{k+1}i_k}\sigma_{i_k} \tag{2}$$

$$\sigma_i - \varrho_{ii_k}\sigma_{i_k} < 0 \qquad \forall i \notin \{i_{k+1}, i_k\} \tag{3}$$

$$-\sigma_{i_k} < \sigma_{i_{k-1}} - \varrho_{i_{k-1}i_k}\sigma_{i_k} \tag{4}$$

$$\sigma_i - \varrho_{ii_k}\sigma_{i_k} < \sigma_{i_{k-1}} - \varrho_{i_{k-1}i_k}\sigma_{i_k} \qquad \forall i \notin \{i_{k-1}, i_k, i_{k+1}.\} \tag{5}$$

Furthermore suppose that $1 - \varrho_{i_{k-1}i_k}\varrho_{i_k i_{k-1}} \neq 0$ *for* $k = 2, \dots, K$ *and that* $\lambda_i > 1$ *for all* i *where* $\lambda_i = \prod_k \nu_{i_k}$ *and the saddle values* ν_{i_k} *are given by*

$$\nu_{i_1} = \frac{\sigma_{i_1}}{\sigma_{i_2} - \varrho_{i_2 i_1}\sigma_{i_1}} \text{ and } \nu_{i_k} = \frac{\varrho_{i_{k-1}i_k}\sigma_{i_k} - \sigma_{i_{k-1}}}{\sigma_{i_{k+1}} - \varrho_{i_{k+1}i_k}\sigma_{i_k}} \text{ for } k > 1.$$

Then for every sufficiently small $\varepsilon > 0$ *the system* (1) *has a stable heteroclinic channel* $V(\iota, \varepsilon)$.

Remark 4. Note that the equations $1 - \varrho_{i_{k-1}i_k}\varrho_{i_k i_{k-1}} = 0$ describe a set of Lebesgue measure zero. Hence with probability one we can assume that the condition $1 - \varrho_{i_{k-1}i_k}\varrho_{i_k i_{k-1}} \neq 0$ is satisfied.

This theorem describes how the system has to be organized for a given stimulus σ to contain a heteroclinic channel by giving a set of coupled inequalities. Suppose that the entries of the coupling matrix ϱ_{ij} are random variables distributed accoding to some distribution $\mathcal{D}_{ij}$. In this paper we want to derive conditions for these probability distributions for the system to have a stable heteroclinic channel under the assumption that the stimulus σ itself is not completely random but its entries are chosen according to specific rules.

Definition 5. *A system modelled by* (1) *is called grossly organized if the nondiagonal entries of the coupling matrix* P *are random variables distributed according to probability distributions* $\mathcal{D}_{ij}$.

The two extreme cases are the following. A system is not organized at all if all nondiagonal entries of the coupling matrix are distributed according to the same probability distribution, in other words the organization of the system contains no information. On the other hand, a system is completely organized if the support of the $\mathcal{D}_{ij}$'s are just points, i.e. the entries are constant.

2. Heteroclinic sequences

Let $\iota = (i_k)$ be an open sequence of equilibria and let I be defined as above. It is easy to show that the conditions (2)–(5) are equivalent to the following conditions

$$0 < \varrho_{i_{k+1}i_k} < \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} \tag{6}$$

$$\frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} < \varrho_{i_{k-1}i_k} < \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + 1 \tag{7}$$

$$\varrho_{i_{k-1}i_k} - \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + \frac{\sigma_i}{\sigma_{i_k}} < \varrho_{ii_k} \qquad \forall i \notin \{i_{k-1}, i_k, i_{k+1}.\} \tag{8}$$

Note that condition (7) ensures that the left hand side of (8) is always greater than one. Furthermore, in the assumptions for Theorem 3 conditions (6) and (7) are coupled by the restraints on the product of the saddle values.

To uncouple these inequalities we choose the following ansatz.

$$\varrho_{i_{k+1}i_k} = \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} - \theta_1$$

$$\varrho_{i_{k-1}i_k} = \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + \theta_2$$

where $\theta_1 \in \left(0, \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}}\right)$ and $\theta_2 \in (0, 1)$. A quick calculation gives $\nu_{i_1} > 1$ and from $\nu_{i_k} = \frac{\theta_2}{\theta_1} > 1$ follows $\lambda_i > 1$ for all $i = 1, \dots, K$. Therefore, when picking $\theta \in (0, 1)$,

$\theta < \min_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}}$ for an arbitrary choice of $\theta_1 \in (0,\theta)$ and $\theta_2 \in (\theta,1)$ the condition on the λ_i is satisfied.

This leads to the following inequalities for $\varrho_{i_{k+1}i_k}, \varrho_{i_{k-1}i_k}$:

$$\frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} - \theta < \varrho_{i_{k+1}i_k} < \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}}$$
$$\frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + \theta < \varrho_{i_{k-1}i_k} < \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + 1.$$

To be able to choose $\varrho_{i_{k+1}i_k}$ and $\varrho_{i_{k-1}i_k}$ from the same interval the following equations have to be satisfied.

$$\frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} = \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + 1$$
$$\frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} - \theta = \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + \theta$$

which is equivalent to $\sigma_{i_{i+1}} = \sigma_{i_k} + \sigma_{i_{k-1}}$ and $\theta = 1/2$. Let us henceforth suppose that the σ_{i_k} form a Fibonacci sequence according to the condition above. To get a uniform bound for all k we take the minimum and the maximum respectively, leading to

$$\max_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} - \frac{1}{2} < \varrho_{i_{k-1}i_k}, \varrho_{i_{k+1}i_k} < \min_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}}$$

It is well known that the fractions of consecutive numbers of a Fibonacci sequence converges to the golden ratio φ, thus the interval is always contained in some ball around zero, independent of the length K of the heteroclinic sequence.

For $i \notin I$ we can choose σ_i to be small enough so that $\max_k \frac{\sigma_i}{\sigma_{i_k}} \leq \max_{i\in I,k} \frac{\sigma_i}{\sigma_{i_k}}$. Under this assumption choosing ϱ_{ii_k} according to

$$\mu_{\mathrm{s}}(K) := \max_{i\in I, k\in\{1\ldots K\}} \frac{\sigma_i}{\sigma_{i_k}} + 1 < \varrho_{ii_k}$$

implies condition (8).

Summarizing the results we obtain

Theorem 6. *Let ι be a sequence of equilibria. Suppose the entries of σ are chosen by the following conditions: the entries corresponding to indices in I form a Fibonnacci sequence with $\sigma_{i_k} < 2\sigma_{i_{k+1}}, k = 1,\ldots,K$, the following inequality holds*

$$\max_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} - \frac{1}{2} < \min_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}},$$

and the rest of the σ_i are chosen small enough. Furthermore suppose that the connectivity matrix (ϱ_{ij}) has ones on the diagonal and the non-diagonal entries are chosen randomly from the following intervals

(1) $\varrho_{i_{k-1}i_k}, \varrho_{i_{k+1}i_k}$ from the interval $\left(\max_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} - \frac{1}{2}, \min_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}}\right)$

(2) $\varrho_{ii_k}, i \notin \{i_{k-1}, i_k, i_{k+1}\}$ from the interval $(\mu_{\mathrm{s}}(K), \infty)$.

Then for every sufficiently small $\varepsilon > 0$ the system (1) has a stable heteroclinic channel $V(\iota,\varepsilon)$.

The theorem gives conditions how a preformatted stimulus σ has to look like for a grossly organized system with a gross structure defined by the two conditions above to contain a heteroclinic channel. A Fibonacci sequence is determined by two initial values but since it can be scaled by an arbitrary positive real number due to the linearity of the recursive formula we can assume $\sigma_{i_1} = 1$. The theorem above only states the conditions for the occurrence of a heteroclinic channel. In order to prove existence we need the following lemma which can be proven by elementary means.

Lemma 7. *Let $J \subset \mathbb{R}$ be an interval and $f, g : J \to \mathbb{R}$ continuous functions. Suppose that there exists $x_0 \in J$ such that $f(x_0) < g(x_0)$. Then there exists a closed rectangle $Q = J_1 \times J_2$ where J_i are closed intervals which is contained in the area that is enclosed by f from below and g from above, given by $A(J, f, g) = \{(x, y) \in J \times \mathbb{R} \mid f(x) < y < g(x)\}$.*

The rectangle contained in the area enclosed by two curves gives us a uniform bound on the ϱ's if the parameter of the Fibonacci sequence is chosen randomly.

Corollary 8. *Let ι be any open sequence of equilibria of length $K \in \mathbb{N}$. Then there are intervals J, J' and $\mu > 0$ such that for any $c \in J, a > 0$ with σ as in Theorem 6 a grossly organized system (1) with coupling matrix chosen given by*

(1) $\varrho_{i_{k-1}i_k}, \varrho_{i_{k+1}i_k}$ picked randomly from the interval J'
(2) $\varrho_{ii_k}, i \notin \{i_{k-1}, i_k, i_{k+1}\}$ picked randomly from the interval (μ, ∞)

has a stable heteroclinic channel $V(\iota, \varepsilon)$ for every sufficiently small $\varepsilon > 0$.

Proof. The functions $f(c) := \max_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} - \frac{1}{2}$ and $g(c) := \min_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}}$ are continuous as the maximum and minimum of continuous functions respectively. For $c = \varphi$, where φ denotes the golden ratio, the fractions $\sigma_{i_k}/\sigma_{i_{k+1}}$ are constantly equal to φ because of the property of the golden ratio. Hence $\min_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} = \varphi = \max_k \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}}$ and therefore $f(\varphi) < g(\varphi)$. Applying Lemma 7 to these functions gives $J' = J_2$ and $J = J_1$. Under these conditions the assumptions of Theorem 6 are satisfied and the claim follows. □

Absolute numbers for the intervals described by the corollary above are depicted in Figure 1.

The restrictions on $\varrho_{i_{k-1}i_k}, \varrho_{i_{k+1}i_k}$ do not depend on K as K tends to infinity because the fraction $\sigma_{i_{k+1}}/\sigma_{i_k}$ converges. Because of the monotonicity of a Fibonacci sequence with positive initial condition the number $\mu_s(K)$ is unbounded for $K \to \infty$.

Numerical simulations have shown that if σ is chosen according to the above results, the i_{K+1}-th node, which is not part of the heteroclinic sequence, is a strongly attractive fixed point. The larger K is the faster it absorbes the orbits so that a typical switching pattern only occurs for small ε.

3. Heteroclinic contours

We now want to consider the case where the sequence of equilibria ι is closed, i.e. we want to consider a heteroclinic contour. We obtain a heteroclinic countour from a heteroclinic sequence by closing it, that is if we expand the conditions of Theorem 3 above to the fist and last node by defining $i_{K+1} = i_1$ and $i_0 = i_K$. Assume that $\sigma_{i_1}, \sigma_{i_2}$ are nonzero. The assumption of a Fibonacci sequence that was derived

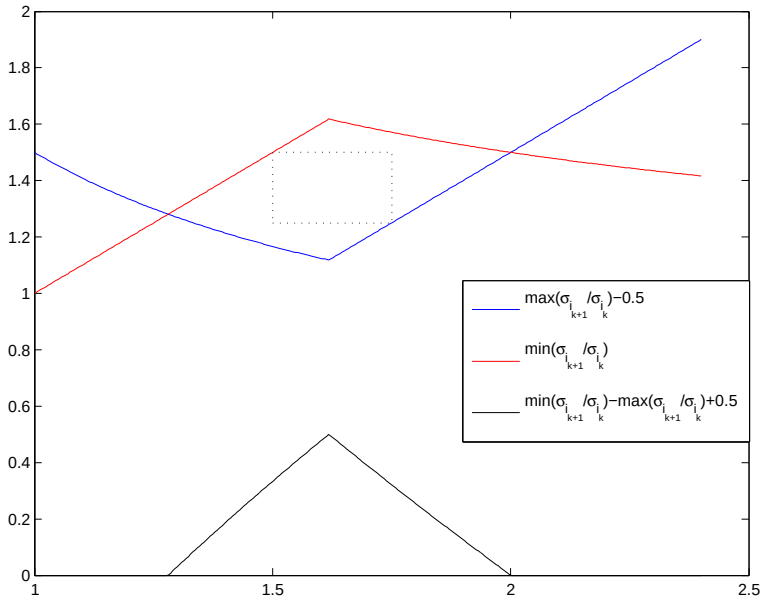

FIGURE 1. Bounds for $\varrho_{i_{k-1}i_k}, \varrho_{i_{k+1}i_k}$ are depicted by the blue and red lines. The dotted rectangle $(1.5, 1.75)\times(1.2, 1.45)$ is an example for a uniform bound for the ϱ's and c.

above however leads to the contradiction $\sigma_{i_1} = \sigma_{i_K} + \sigma_{i_{K-1}}$ when applied to a heteroclinic contour.

Therefore a new ansatz is needed. Note that the inequality $\sigma_{i_k}\theta < \sigma_{i_{k+1}}$ derived above suggests a restricted geometric growth. Furthermore we will let go of the condition that led to the occurrence of the Fibonacci sequence in the first place, namely to be able to draw $\varrho_{i_{k-1}i_k}, \varrho_{i_{k+1}i_k}$ from the same interval. Suppose that $\sigma_{i_k} = ac^{k-1}$. The case $c = 1$ represents the case of all σ_{i_k} being equal which was already considered in [ARV04]. Under these assumptions conditions (6)–(8) for a heteroclinic contour are given by the following inequalities

$$0 < \varrho_{i_{k+1}i_k} < \frac{\sigma_{i_{k+1}}}{\sigma_{i_k}} \qquad \text{for } k \neq K \tag{9}$$

$$0 < \varrho_{i_1 i_K} < \frac{\sigma_{i_1}}{\sigma_{i_K}} \tag{10}$$

$$\frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} < \varrho_{i_{k-1}i_k} < \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + 1 \qquad \text{for } k \neq 1 \tag{11}$$

$$\frac{\sigma_{i_K}}{\sigma_{i_1}} < \varrho_{i_K i_1} < \frac{\sigma_{i_K}}{\sigma_{i_1}} + 1 \tag{12}$$

$$\varrho_{i_{k-1}i_k} - \frac{\sigma_{i_{k-1}}}{\sigma_{i_k}} + \frac{\sigma_i}{\sigma_{i_k}} < \varrho_{ii_k} \qquad \forall i \notin \{i_{k-1}, i_k, i_{k+1}.\} \tag{13}$$

As in the previous section we can uncouple the first four inequalities by choosing θ_1, θ_2. The saddle values are $\nu_{i_k} = \theta_2/\theta_1$ for all $k = 1, \ldots, K$ and thus choose $\theta \in (0, \min\{1, c\})$ to obtain the following inequalities. The ratio of numbers from a

geometric sequence can be replaced by the corresponding powers of c.

$$c - \theta < \varrho_{i_{k+1}i_k} < c \qquad \text{for } k \neq K \tag{14}$$

$$c^{1-K} - \theta < \varrho_{i_1 i_K} < c^{1-K} \tag{15}$$

$$c^{-1} + \theta < \varrho_{i_{k-1}i_k} < c^{-1} + 1 \qquad \text{for } k \neq 1 \tag{16}$$

$$c^{K-1} + \theta < \varrho_{i_K i_1} < c^{K-1} + 1 \tag{17}$$

Therefore θ must satisfy the restraints $\theta \in \left(0, \min\{c^{K-1}, c\}\right)$. In order to draw $\varrho_{i_{k+1}i_k}, \varrho_{i_1 i_K}$ and $\varrho_{i_{k-1}i_k}, \varrho_{i_K i_1}$ from the same interval respectively define

$$f_l(c,\theta) = \begin{cases} c^{1-K} - \theta & 0 < c < 1 \\ c - \theta & 1 \leq c \end{cases}$$

$$f_u(c,\theta) = \begin{cases} c & 0 < c < 1 \\ c^{1-K} & 1 \leq c \end{cases}$$

$$g_l(c,\theta) = \begin{cases} c^{-1} + \theta & 0 < c < 1 \\ c^{K-1} + \theta & 1 \leq c \end{cases}$$

$$g_u(c,\theta) = \begin{cases} c^{K-1} + 1 & 0 < c < 1 \\ c^{-1} + 1 & 1 \leq c \end{cases}$$

which describe the boundaries of the intervals.

For $i \notin I = \{i_k \mid k = 1, \dots, K\}$ we can choose σ_i to be small enough so that $\max_k \frac{\sigma_i}{\sigma_{i_k}} \leq \max\{1, c^{K-1}, c^{1-K}\}$. With this assumption choosing ϱ_{ii_k} according to

$$\mu_c(K) := \max\{1, c^{K-1}, c^{1-K}\} + 1 < \varrho_{ii_k}$$

implies condition (13).

From the calculations above we obtain the following result.

Theorem 9. *Let ι be a closed sequence of equilibria. There exists a suitable neighborhood U of $1 \in \mathbb{R}$ such that the following statements hold: Suppose that the components of σ are such that $\sigma_{i_k} = ac^{k-1}$ with $c \in U, a > 0$ and σ_i small enough for $i \notin I$. Let $\theta \in \left(0, \min\{c^{K-1}, c\}\right)$. Furthermore suppose that the coupling matrix (ϱ_{ij}) has ones on the diagonal and the non diagonal entries are chosen randomly from the follwing intervals:*

(1) $\varrho_{i_{k+1}i_k}, \varrho_{i_1 i_K}$ from the interval $(f_l(c,\theta), f_u(c,\theta)) \neq \emptyset$,
(2) $\varrho_{i_{k-1}i_k}, \varrho_{i_K i_1}$ from the interval $(g_l(c,\theta), g_u(c,\theta)) \neq \emptyset$,
(3) $\varrho_{ii_k}, i \notin \{i_{k-1}, i_k, i_{k+1}\}$ from the interval $(\mu_c(K), \infty)$.

Then for every sufficiently small $\varepsilon > 0$ the system (1) has a stable heteroclinic channel $V(\iota, \varepsilon)$.

Proof. The only thing that remains to be shown is the existence of such a neighborhood. Note that the functions defined above are continuous at 1 and that we have $f_l(\theta, 1) = 1 - \theta < 1 = f_u(\theta, 1)$ and $g_l(\theta, 1) = 1 + \theta < 2 = g_u(\theta, 1)$. Because of continuity there is for any fixed $\theta \in \left(0, \min\{c^{K-1}, c\}\right)$ a whole neighborhood U of 1 such that $f_l(\theta, c) < f_u(\theta, c)$ and $g_l(\theta, c) < g_u(\theta, c)$ for every $c \in U$. □

The theorem states that if we fix a suitable c then we can find a heteroclinic channel in the system modeled by (1) with coupling according to the theorem. It is

desireable though to have a result that states that for any c chosen randomly from some interval and any grossly organized system (that may depend on the interval from which c is chosen from) there exists a stable heteroclinic channel.

Corollary 10. *Let $\theta = \frac{1}{2}$ and ι be any closed sequence of equilibria of length $K \in \mathbb{N}$. Then there are intervals J, J', J'' and $\mu > 0$ such that for any $c \in J, a > 0$ with σ as in Theorem 9 a grossly organized system* (1) *with coupling matrix chosen given by*

1. *$\varrho_{i_{k+1}i_k}, \varrho_{i_1 i_K}$ picked randomly from the interval J'*
2. *$\varrho_{i_{k-1}i_k}, \varrho_{i_K i_1}$ picked randomly from the interval J''*
3. *$\varrho_{ii_k}, i \notin \{i_{k-1}, i_k, i_{k+1}\}$ picked randomly from the interval (μ, ∞)*

has a stable heteroclinic channel $V(\iota, \varepsilon)$ for every sufficiently small $\varepsilon > 0$.

Proof. Consider the functions $h_u(c,\theta) = g_u(c,\theta) + (f_u(1,\theta) - g_u(1,\theta))$ and $h_l(c,\theta) = g_l(c,\theta) + (f_l(1,\theta) - g_l(1,\theta))$. Note that $h_l(1,\theta) = f_l(1,\theta)$ and $h_u(1,\theta) = f_u(1,\theta)$. Define

$$\psi_1(c,\theta) = \min\{f_u(c,\theta), h_u(c,\theta\}, \quad \psi_2(c,\theta) = \max\{f_l(c,\theta), h_l(c,\theta\}.$$

The functions ψ_i are continuous and $\psi_2(1,\theta) < \psi_1(1,\theta)$. Applying Lemma 7 provides us with intervals J_1, J_2. By construction and as in the proof of Corollary 8 there is $\kappa > 0$ and μ large enough such that with $J = J_1, J' = J_2, J'' = \kappa + J_2 = \{\kappa + x \mid x \in J_2\}$ the assumptions of Theorem 9 are satisfied. □

Absolute numbers for the rectangles described in the corollary above are depicted in Figure 2. Furthermore we simulated a system that is set up according to the results of Corollary 10 in Figure 3.

Define $c_{\min}, c_{\max}$ to be the smallest and largest values of c for a given θ such that the assupmtions of Theorem 9 are still satisfied. With increasing length of the heteroclinic sequence the difference $c_{\max} - c_{\min}$ gets smaller, cf. Figure 4. Therefore as K gets large c must be chosen very close to one to get the desired behavior. Hence the variability in the setup is largest when K is small. In contrast to the situation of heteroclinic contours the number $\mu_c(K)$ that bounds the remaining elements of the connectivity matrix away from zero is not diverging since we keep c artificially small by requiring that $\varrho_{i_1 i_K}$ can be drawn as $\varrho_{i_{k+1}i_k}$ and $\varrho_{i_K i_1}$ from the same interval as $\varrho_{i_{k-1}i_k}$. Lifting these requirements allowing c to be bounded away from 1 and we get the same divergent behavior $\mu_c(K) \to \infty$ for $K \to \infty$ as in the case of heteroclinic sequences discussed in Section 2.

4. Conclusion and Outlook

In this paper we have shown that it is possible to find stable heteroclinic sequences/contours in systems that are grossly organized. In the case of open sequences two different probability distributions were sufficient to get a stable heteroclinic channel and three in the case of a contour assuming that the stimulus σ was organized according to a certain scheme with random parameters. Note that the conditions derived here are just extreme cases. By putting more restraints on the coupling strengths one can obtain a stable heteroclinic channel under a wider range of stimuli. However under these conditions we have obtained an overall coupling scheme.

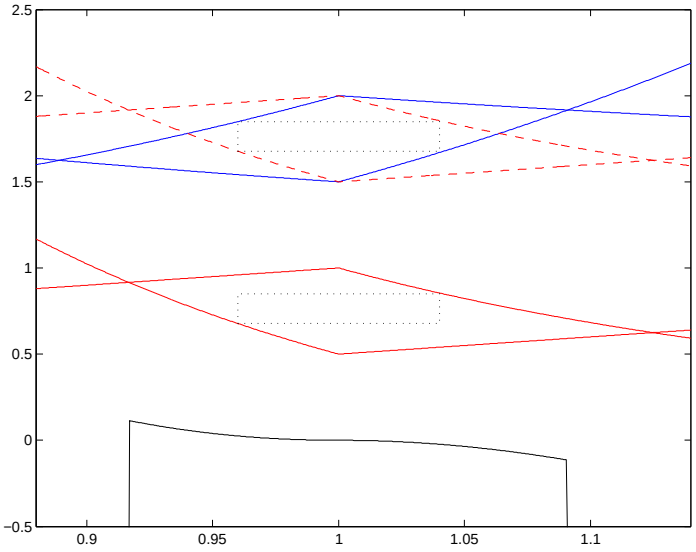

FIGURE 2. For $K = 5$ and $\theta = 0.5$: Functions f_l, f_u are depicted in red and g_l, g_u in blue which describe the bounds for $\varrho_{i_{k-1}i_k}, \varrho_{i_{k+1}i_k}$ resp. The dashed red line is the graphs of $f_l + 1, f_u + 1$. The black line depics the difference in interval length in the allowed regime. The dotted rectangle $(0.96, 1.04) \times (1.677, 1.849)$ and the corresponding vertical translate are examples for uniform bounds for the ϱ's and c.

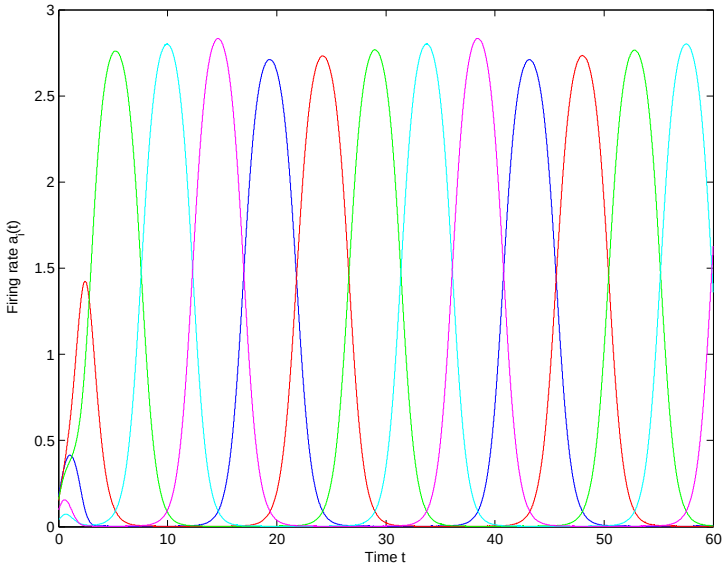

FIGURE 3. Simulation with the conditions shown in Figure 2. Different colors describe different a_i's.

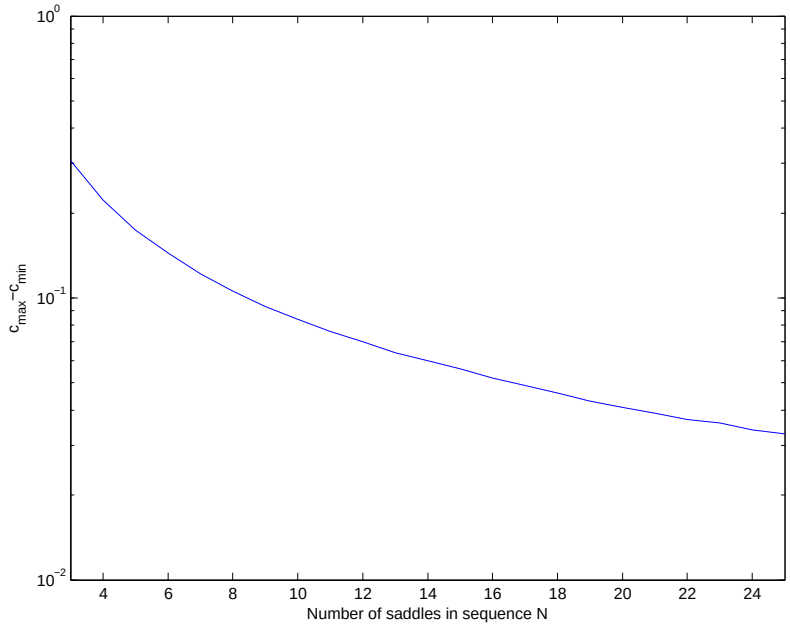

FIGURE 4. The maximum interval the parameter c can be chosen from depending on the number of saddles in the heteroclinic contour.

Since the theorems proved above specify intervals for the parameters of the LV equations one can say that if the distributions are uniform on these interval a heteroclinic channel will occur with probability one. If the entries of P are to be taken from some other distribution, one can use the results to calculate a lower bound for the probability that a stable heteroclinic channel will occur.

An interesting observation that will require further investigation is the fact that $\mu_{\mathrm{s}}(K), \mu_{\mathrm{c}}(K) \to \infty$ as $K \to \infty$. This provides evidence that in a system that occurs in nature, where coupling strengths are generally bounded, the number of nodes in occuring heteroclinic sequences is bounded as well.

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