Nonlinear Dynamics of Neural Firing

BENG/BGGN 260 Neurodynamics

University of California, San Diego

Week 3

Reading Materials

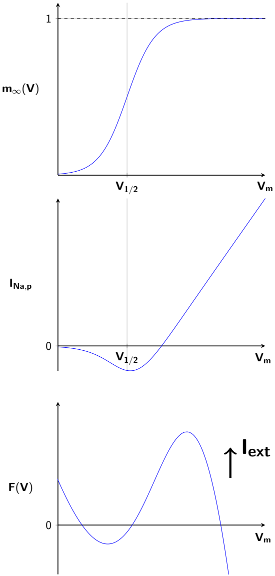
- E.M. Izhikevich, *Dynamical Systems in Neuroscience*, MIT Press, 2007, Ch. 1-2, pp. 53-121.
- C. Koch, *Biophysics of Computation*, Oxford Univ. Press, 1999, Ch. 7, pp. 172-192.

$$C_m \frac{dV_m}{dt} = I_{ext} - \underbrace{\bar{g}_{Na} m_\infty (V_m)(V_m - E_{Na})}_{} - g_L (V_m - E_L)$$

$I_{Na,p}$ persistent, fast Na^+ ($Izh.\ p.55$)

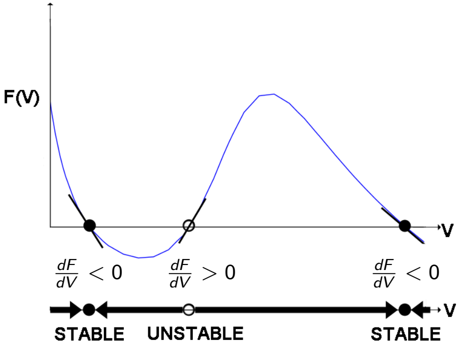
$$m_\infty = \frac{1}{1 + e^{-\left(V_m - V_{1/2}\right)/k}}$$

$$\Rightarrow \frac{dV}{dt} = F(V)$$



Phase Portrait, Equilibria

$$\frac{dV}{dt} = F(V)$$



Small-signal analysis:

$$V = V_0 + \widetilde{v}$$

$$\frac{d\widetilde{v}}{dt} \approx \alpha\widetilde{v} \quad \text{with} \quad \alpha = \left.\frac{dF}{dV}\right|_{V_0}$$

$$\widetilde{v} \approx \widetilde{v}_0 e^{+\alpha t} \quad \text{stable for} \quad \alpha \leq 0 \text{ (decaying perturbation)}$$

Bistability and Hysteresis

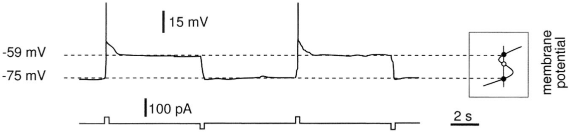


Figure 3.16: Membrane potential bistability in a cat TC neuron in the presence of ZD7288 (pharmacological blocker of I_b). (Modified from Fig. 6B of Hughes et. al. 1999).

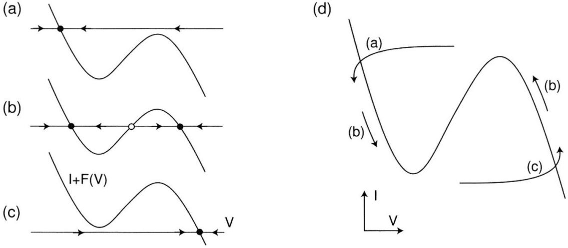


Figure 3.17: Bistability and hysteresis loop as I changes.

Izhikevich, pg. 66

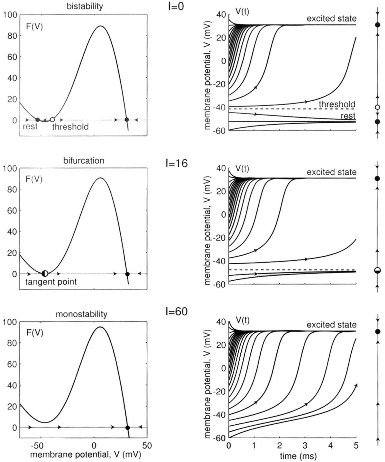


Figure 3.25: Bifurcation in the $I_{Na,p}$-model (3.5): The resting state and the threshold state coalesce and disappear when the parameter I increases.

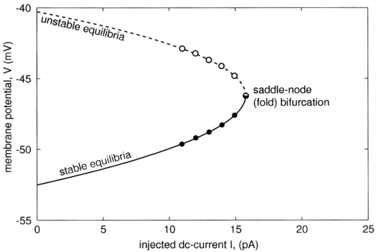


Figure 3.30 Bifurcation diagram of the system in Fig. 3.26.

Izhikevich, pg. 72, pg. 77

Two-Dimensional Systems

- **Morris-Lecar**

$$C_m \frac{dV_m}{dt} = I_{ext} - \bar{g}_K \underset{\substack{\downarrow \\ n \quad (n = w) \\ \downarrow \\ n^4}}{w} (V - E_K) - \bar{g}_{\underset{\substack{\downarrow \\ Na}}{Ca}} \underset{\substack{\big\downarrow \\ {m_\infty}^3 (1 - n)}}{m_\infty (V)} (V - E_{\underset{\substack{\downarrow \\ Na}}{Ca}}) - g_L (V - E_L) \quad \begin{array}{c} \text{ML} \\ \downarrow \\ \to I_{Na,p} + I_K \\ \downarrow \\ \text{simplified HH} \end{array}$$

$$\frac{dw}{dt} = \frac{w_\infty (V_m) - w}{\tau_w (V_m)}$$

- **FitzHugh-Nagumo, etc.**

$$\Rightarrow \begin{cases} \frac{dV}{dt} & = F(V, W) \\ \frac{dW}{dt} & = G(V, W) \end{cases}$$

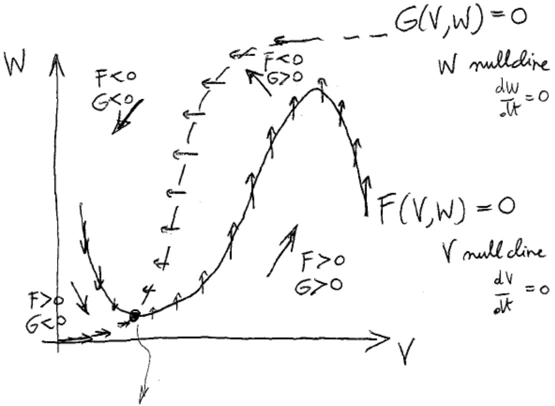
Dynamic repertoire:

- **Stationary Points**
- **Limit Cycles**
- **Stable Attractors**

No other limits in 2-D (since V and W are bounded)

Nullclines

$$\begin{cases} \frac{dV}{dt} & = F(V, W) \\ \frac{dW}{dt} & = G(V, W) \end{cases}$$



Equilibrium

- Stable $\Rightarrow$ Stationary Point
- Unstable?

Stability

Equilibria:

$$\begin{cases} F\left(V_0, W_0\right) & = 0 \\ G\left(V_0, W_0\right) & = 0 \end{cases} \quad \text{intersection of nullclines}$$

Linear Analysis:

$$\begin{cases} V & = V_0 + \tilde{v} \\ W & = W_0 + \tilde{w} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d\tilde{v}}{dt} & = a\tilde{v} + b\tilde{w} \\ \frac{d\tilde{w}}{dt} & = c\tilde{v} + d\tilde{w} \end{cases} \quad \text{with} \quad \begin{array}{ll} a = \left.\frac{\partial F}{\partial V}\right|_{V_0, W_0} & b = \left.\frac{\partial F}{\partial W}\right|_{V_0, W_0} \\ c = \left.\frac{\partial G}{\partial V}\right|_{V_0, W_0} & d = \left.\frac{\partial G}{\partial W}\right|_{V_0, W_0} \end{array}$$

Eigenanalysis:

$$\mathbf{U} = \begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix} \quad \overset{\text{Jacobian}}{\mathbf{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{d\mathbf{U}}{dt} = \mathbf{A}\,\mathbf{U}, \text{ or } \mathbf{U} = c_1 \mathbf{U}_1 e^{\lambda_1 t} + c_2 \mathbf{U}_2 e^{\lambda_2 t} \quad \text{with} \quad \begin{array}{ll} \mathbf{A}\,\overset{\text{eigenvectors}}{\mathbf{U}_1} & = \overset{\text{eigenvalues}}{\lambda_1}\mathbf{U}_1 \\ \mathbf{A}\,\mathbf{U}_2 & = \lambda_2 \mathbf{U}_2 \end{array}$$

$$\begin{cases} Re\,\lambda_1 < 0 \text{ and } Re\,\lambda_2 < 0 & \Rightarrow \text{stable fixed point} \\ Re\,\lambda_1 > 0 \text{ or } Re\,\lambda_2 > 0 & \Rightarrow \text{unstable} \end{cases}$$

Stability (Continued)

Eigenanalysis:

$$\mathbf{A}\,\mathbf{U} = \lambda\mathbf{U}$$

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \tau\lambda + \Delta = 0 \qquad \text{with} \qquad \begin{cases} \tau & = a+d \quad : \quad Trace\left(\mathbf{A}\right) \\ \Delta & = ad-bc \quad : \quad det\left(\mathbf{A}\right) \end{cases}$$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

$$\Rightarrow \begin{cases} \tau^2 & > 4\Delta : \ \text{real, distinct eigenvalues} \\ \tau^2 & = 4\Delta : \ \text{real, repeated eigenvalues} \\ \tau^2 & < 4\Delta : \ \text{complex conjugate eigenvalues} \end{cases}$$

Stability of 2-D Equilibria

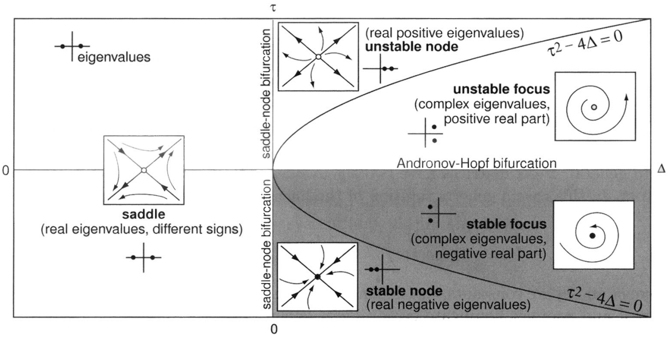


Figure 4.15: Classification of equilibria according to the trace (τ) and the determinant (Δ) of the Jacobian matrix $\mathbf{A}$. The shaded region corresponds to stable equilibria.

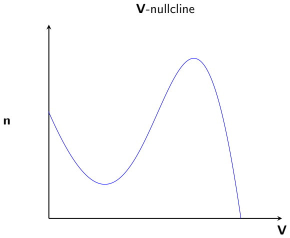
Izhikevich, pg. 104

Persistent sodium and potassium $\left(I_{Na,p} + I_K\right)$ is equivalent to Morris-Lecar $(n = w)$.

V − nullcline :

$$F(V,n) = \frac{1}{C}\Big(I_{ext} - \bar{g}_K n (V - E_K) - \bar{g}_{Na} m_\infty (V)(V - E_{Na}) - g_L (V - E_L)\Big) = 0 \Rightarrow$$

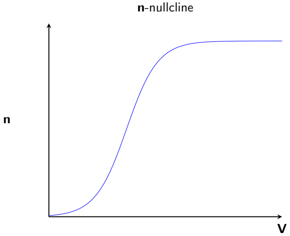
$$n = \frac{I_{ext} - \bar{g}_{Na} m_\infty (V)(V - E_{Na}) - g_L (V - E_L)}{\bar{g}_K (V - E_K)}$$



n − nullcline :

$$G(V,n) = \frac{n_\infty (V) - n}{\tau_n (V)} = 0 \Rightarrow$$

$$n = n_\infty (V)$$



$I_{Na,p} + I_K$ (or ML) Nullclines

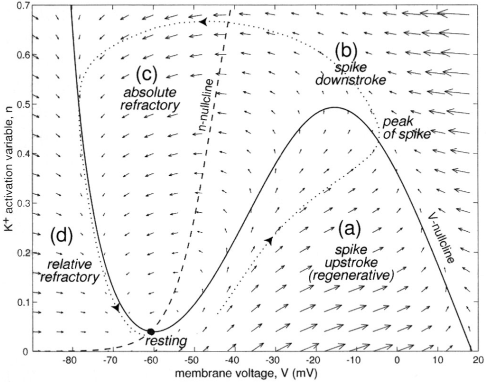


Figure 4.4: Nullclines of the $I_{Na,p} + I_K$-model (4.1, 4.2) with low-threshold K^+current in Fig. 4.1b. (The vector field is slightly distorted for the sake of clarity of illustration).

Izhikevich, pg. 93

Graded Action Potentials

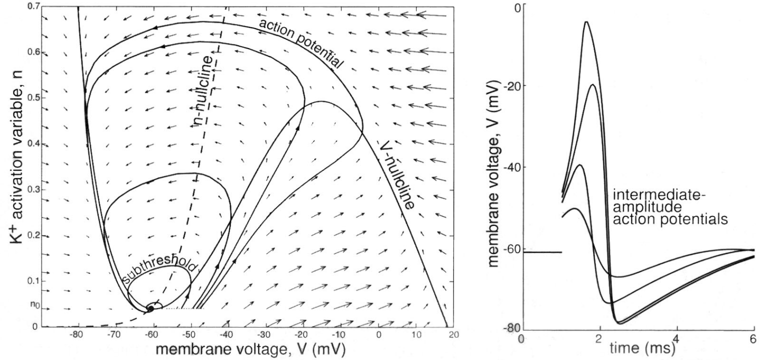


Figure 4.7: Failure to generate all-or-none action potentials in the $I_{Na,p} + I_K$-model.

Izhikevich, pg. 95

Variable Threshold

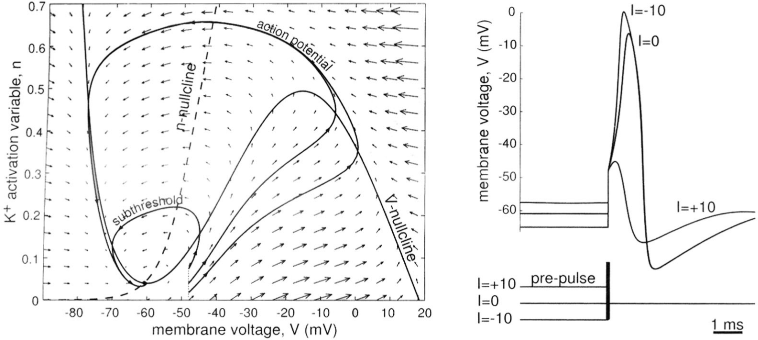


Figure 4.8: Failure to have a fixed value of threshold voltage in the $I_{Na,p} + I_K$-model.

Izhikevich, pg. 96

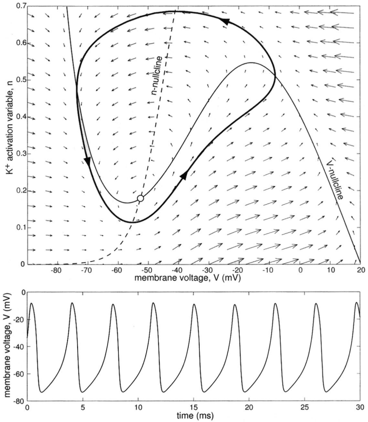


Figure 4.10: Stable limit cycle in the $I_{Na,p} + I_K$-model (4.1, 4.2) with low-threshold K^+current and $I = 40$.

Izhikevich, pg. 97.