

BENG 186B Bode Plot Guide

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This guide serves as an introduction to finding magnitude and phase of transfer functions, as well as making Bode plots, which you may see throughout the class. Full length examples can be found at the end of the guide.

I. Bode Plot Introduction

Bode plots give engineers a way to visualize the effect of their circuit, in terms of voltage magnitude and phase angle (shift). A Bode plot consists of two separate plots, one for magnitude and one for phase angle.

II. Magnitude

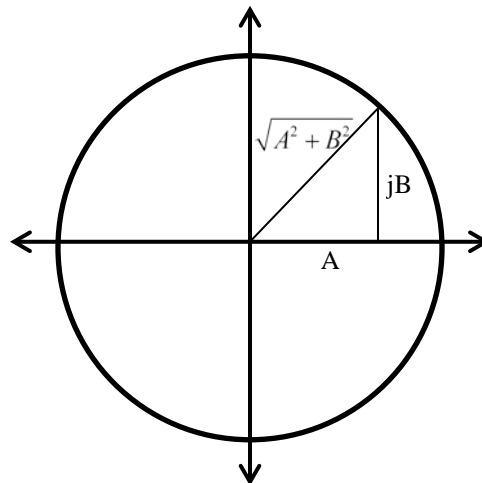
The first part of making a Bode plot is finding the magnitude of the transfer function. Remember that the transfer function is the “black box” of your circuit which changes the voltage input into the voltage output:

$$H(j\omega) = \frac{V_{out}}{V_{in}} \quad (1)$$

Often times, the transfer function $H(j\omega)$ has both real and imaginary components, due to frequency dependent impedances, such as the general form $A + jB$. Just as it sounds, finding the magnitude follows the form:

$$|H(j\omega)| = \sqrt{A^2 + B^2} \quad (2)$$

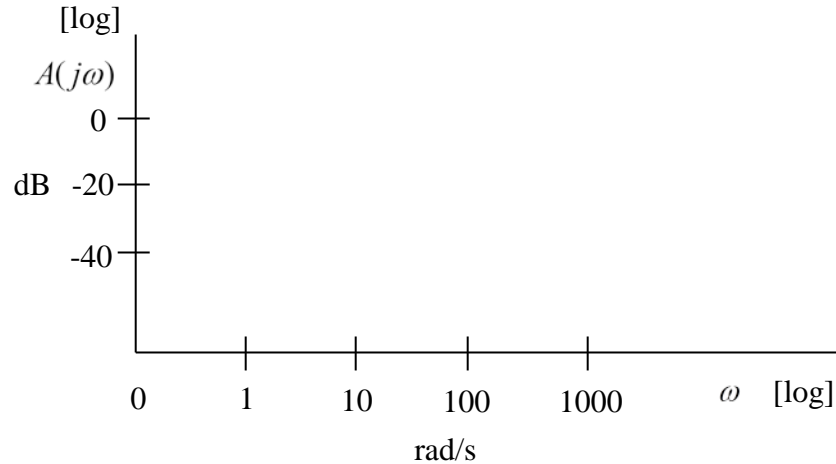
One way of visualizing magnitude is by a polar coordinate sketch.



For the purposes of making a Bode plot, log-normalized amplitude is used on the y-axis of the magnitude plot. This makes amplitude in units of dB (decibels).

$$A(j\omega) = 20\log_{10}(|H(j\omega)|) \quad (3)$$

A log scale is also used for radial frequency (ω) on the x-axis. This yields the following plot format:



For more complex transfer functions, follow these steps:

- 1) Determine the transfer function of the system.

$$H(s) = \frac{(s + z_1)(s + z_2)}{s(s + p_1)} \quad (4)$$

Where the poles are at $s = 0$ and $s = -p_1$, and the zeros are at $s = -z_1$ and $s = -z_2$ in the s domain. In the frequency domain, these represent the break frequencies at which the plot is $\pm 3\text{dB}$.

Note: In the s domain, roots of the numerator are called **zeros** and roots of the denominator are called **poles**.

- 2) Convert your transfer function into the frequency domain by substituting $j\omega$ for s . Rewrite both the numerator and denominator into standard form.

$$H(j\omega) = \frac{z_1 z_2 (j\omega / z_1 + 1)(j\omega / z_2 + 1)}{p_1 j\omega (j\omega / p_1 + 1)} \quad (5)$$

- 3) Find magnitude of the transfer function using Eqn. (2)

$$A(\omega) = |H(j\omega)| = \frac{|z_1||z_2| \left(\sqrt{(\omega/z_1)^2 + 1} \right) \left(\sqrt{(\omega/z_2)^2 + 1} \right)}{|p_1| \omega \left(\sqrt{(\omega/p_1)^2 + 1} \right)} \quad (6)$$

Hint: remember that the magnitude of the ratio (V_{out}/V_{in}) is the ratio of the magnitudes. This means that:

$$|H(j\omega)| = \left| \frac{Num}{Denom} \right| = \frac{|Num|}{|Denom|} \quad (7)$$

To convert this to units of dB for making your Bode plot:

$$A(\omega) \text{ in dB} = 20 \log_{10} |H(j\omega)| = 20 \left(\begin{aligned} &\log_{10} \left(\frac{|z_1||z_2|}{|p_1|} \right) + \log_{10} \left(\sqrt{(\omega/z_1)^2 + 1} \right) + \log_{10} \left(\sqrt{(\omega/z_2)^2 + 1} \right) - \log_{10}(\omega) - \\ &\log_{10} \left(\sqrt{(\omega/p_1)^2 + 1} \right) \end{aligned} \right) \quad (8)$$

Be sure to remember your log rules:

$$\log(AB) = \log(A) + \log(B) \quad (9)$$

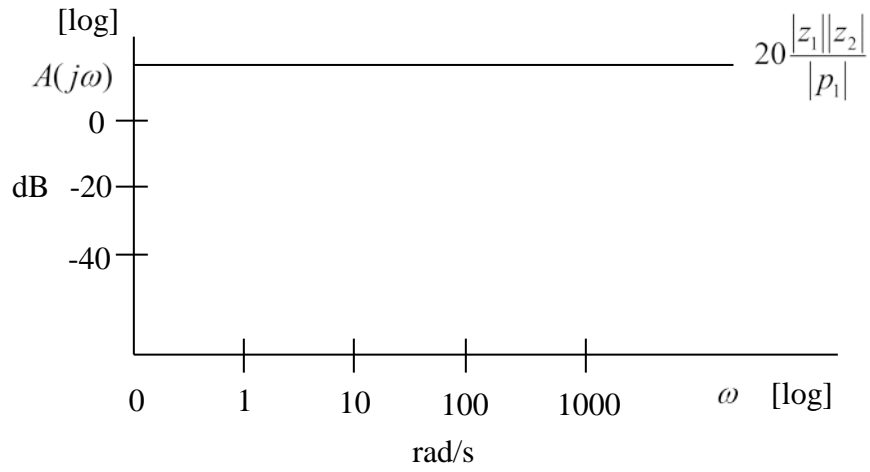
$$\log\left(\frac{A}{B}\right) = \log(A) - \log(B) \quad (10)$$

Notice that when $\omega \rightarrow 0$, the 2nd, 3rd, and 5th terms of Eqn. (8) become zero. On a log-log magnitude plot, these terms becomes a flat line through zero.

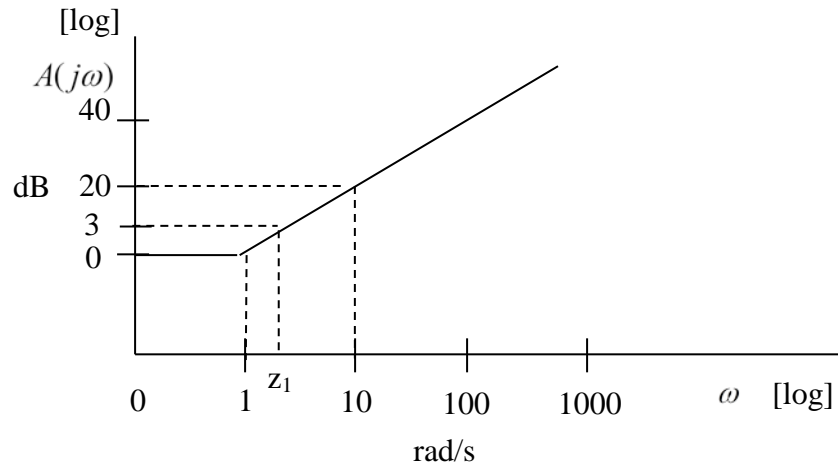
When $\omega \rightarrow \infty$, the 2nd, 3rd, and 5th terms of Eqn. (8) become $\log_{10}\left(\frac{\omega}{z_1}\right)$, $\log_{10}\left(\frac{\omega}{z_2}\right)$, and $\log_{10}\left(\frac{\omega}{p_1}\right)$ respectively. On a log-log magnitude plot, these terms becomes a line with a slope of 1.

If you were to plot magnitude curve, each of the zeros contributes an upward “kink”, while each of the poles contributes a downward “kink” on the frequency (ω) axis. Let’s take a look at contribution to the curve from each of the term in Eqn. 8 (Note: the break frequencies for the poles/zeros are randomly chosen for the purpose of the plot example):

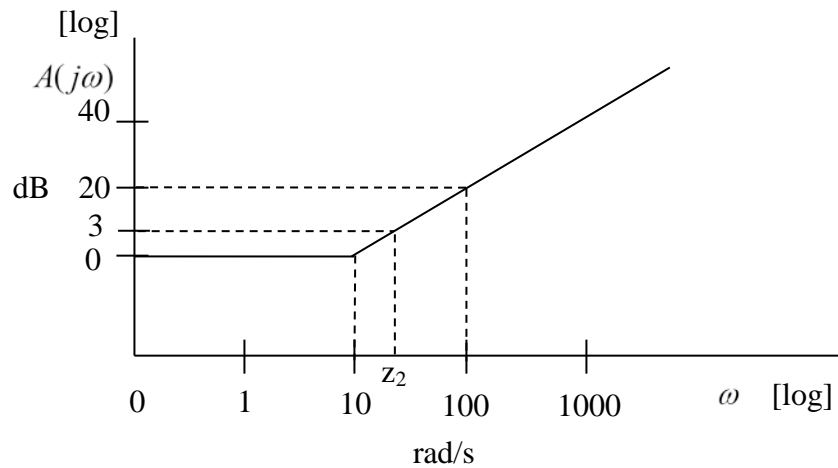
Term (constant): $20\log_{10}\left(\frac{|z_1||z_2|}{|p_1|}\right)$



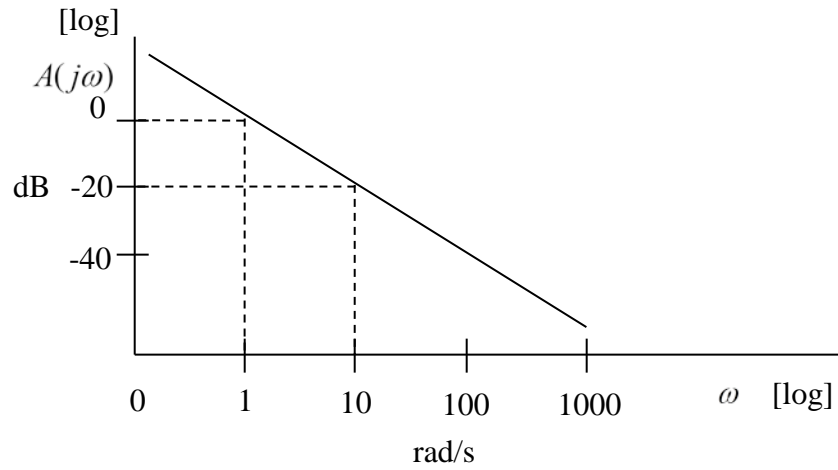
Term (zero): $20\log_{10}\left(\sqrt{(\omega/z_1)^2 + 1}\right)$



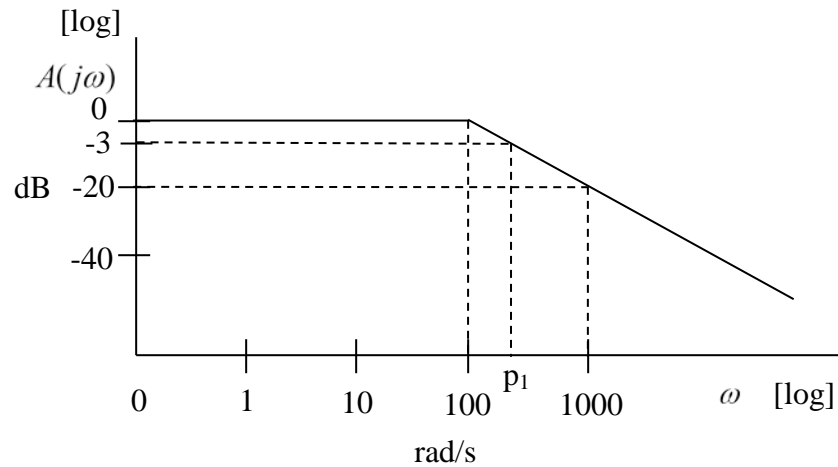
Term (zero): $20\log_{10}\left(\sqrt{(\omega/z_2)^2 + 1}\right)$



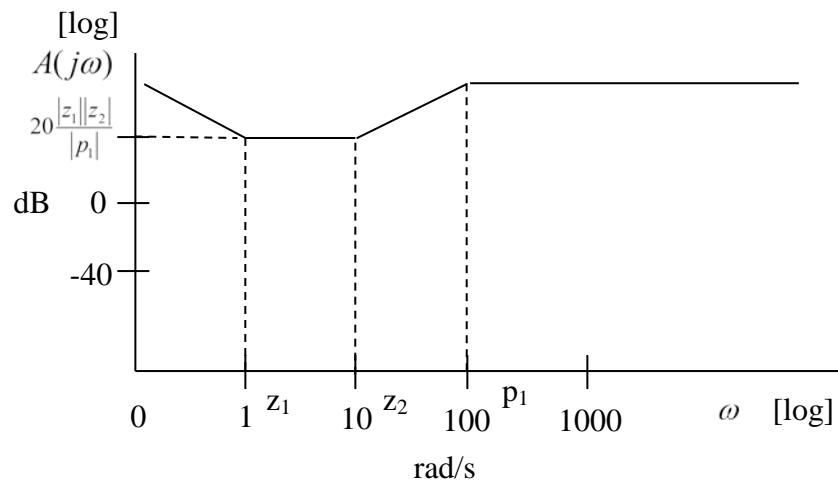
Term (pole): $-20\log_{10}(\omega)$



Term (pole): $-20\log_{10}\left(\sqrt{(\omega/p_1)^2 + 1}\right)$



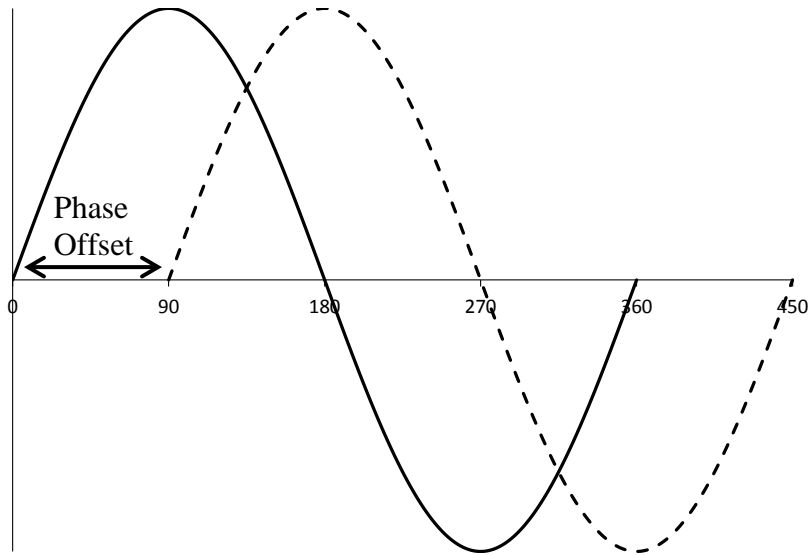
Now, using superposition, we can sum the contributions from each of the magnitude terms (previous 5 plots), yielding the overall magnitude plot.



To refine your plot, you may smooth the boundaries between poles/zeros for a more realistic plot.

III. Phase Angle

The second part of making a Bode plot is finding the phase angle of the transfer function. The phase angle can be thought of as the offset in time (degrees), as in the following sine wave example:



Finding Phase Angle

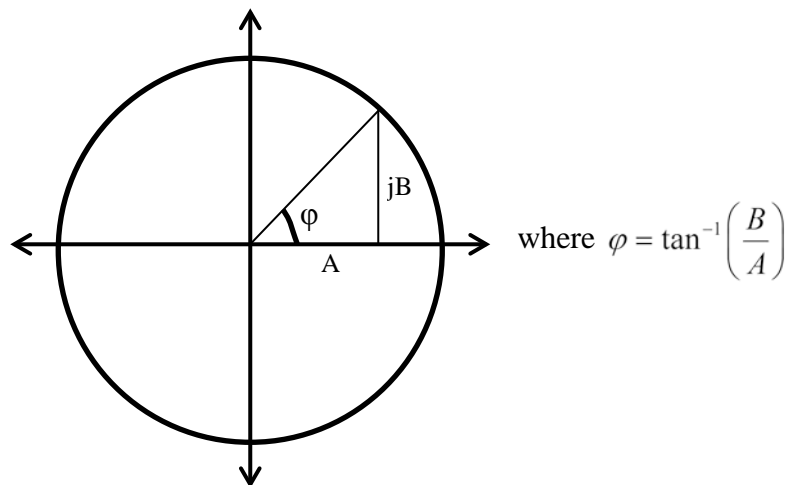
The phase angle is denoted by the symbol \angle . For simple functions, phase is simply:

$$\angle H(j\omega) = \angle \frac{V_{out}}{V_{in}} \quad (11)$$

By the definition of phase, the following statement can be made:

$$\angle \frac{V_{out}}{V_{in}} = \tan^{-1} \left(\frac{\text{Imaginary}[H(j\omega)]}{\text{Real}[H(j\omega)]} \right) \quad (12)$$

You can visualize phase from arctan by the polar coordinate sketch.

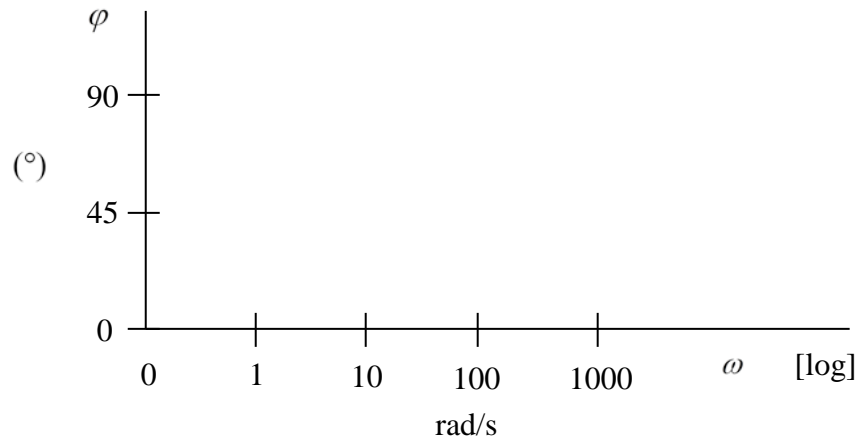


For more complex functions, you can use the definition of phase to calculate the phase of the numerator and denominator separately.

$$\angle \frac{V_{out}}{V_{in}} = \frac{\angle V_{out}}{\angle V_{in}} = \angle V_{out} - \angle V_{in} \quad (13)$$

To determine phase angle of each term, simply use the arctan, Eqn. (12).

A log scale is also used for radial frequency (ω) on the x-axis. The y-axis is phase angle in degrees (linear scale, not log). This yields the following plot format:



For more complex transfer functions, use the following steps to plot phase. Let's use the same transfer function as in the magnitude plot.

$$H(j\omega) = \frac{z_1 z_2 (j\omega / z_1 + 1)(j\omega / z_2 + 1)}{p_1 j\omega (j\omega / p_1 + 1)} \quad (14)$$

Use Eqn. (11) and (13) to find phase angle.

$$\angle H(j\omega) = \angle \frac{z_1 z_2 (j\omega / z_1 + 1)(j\omega / z_2 + 1)}{p_1 j\omega (j\omega / p_1 + 1)} = \frac{\angle z_1 z_2 (j\omega / z_1 + 1)(j\omega / z_2 + 1)}{\angle p_1 j\omega (j\omega / p_1 + 1)} \quad (15)$$

Separate terms using multiplication and division rules

$$\angle H(j\omega) = \angle \frac{z_1 z_2}{p_1} + \angle(j\omega / z_1 + 1) + \angle(j\omega / z_2 + 1) - \angle j\omega - \angle(j\omega / p_1 + 1) \quad (16)$$

Use Eqn. (12) to solve for the angle of each term

$$\angle H(j\omega) = \tan^{-1}\left(\frac{0}{z_1 z_2 / p_1}\right) + \tan^{-1}\left(\frac{\omega / z_1}{1}\right) + \tan^{-1}\left(\frac{\omega / z_2}{1}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega / p_1}{1}\right) \quad (17)$$

The 1st and 4th terms can then be simplified

$$\angle H(j\omega) = 0^\circ + \tan^{-1}\left(\frac{\omega / z_1}{1}\right) + \tan^{-1}\left(\frac{\omega / z_2}{1}\right) - 90^\circ - \tan^{-1}\left(\frac{\omega / p_1}{1}\right) \quad (18)$$

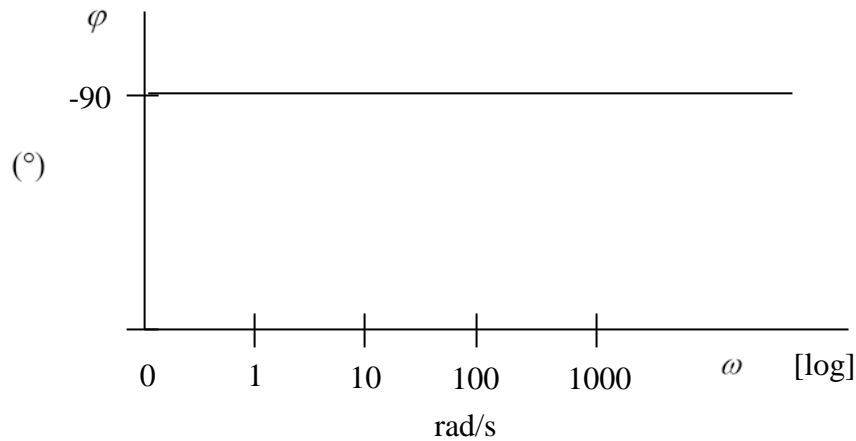
$$\angle H(j\omega) = -90^\circ + \tan^{-1}(\omega / z_1) + \tan^{-1}(\omega / z_2) - \tan^{-1}(\omega / p_1) \quad (19)$$

Notice that when $\omega \rightarrow 0$, the 2nd, 3rd, and 4th terms of Eqn. (19) become zero.

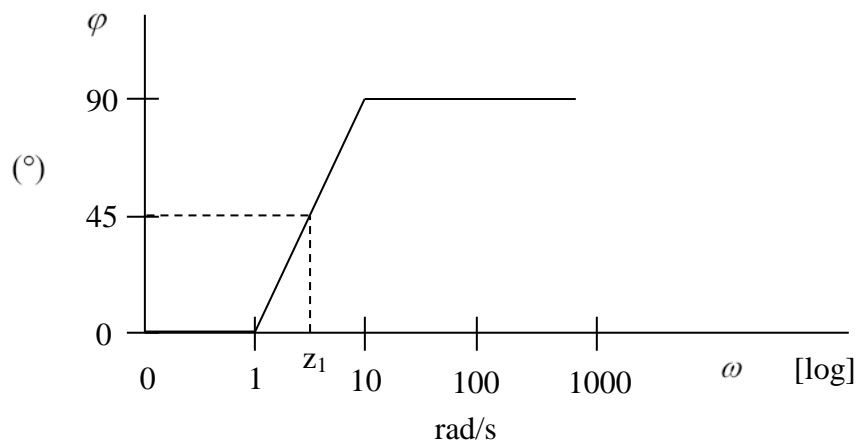
When $\omega \rightarrow \infty$, the 2nd, 3rd, and 4th terms of Eqn. (19) become 90° , 90° , and -90° respectively.

If you were to plot the phase curve, each of the zeros contributes a $+90^\circ$ offset, while each of the poles contributes a -90° offset. Let's take a look at contribution to the curve from each of the terms in Eqn. 19 (Note: the break frequencies for the poles/zeros are randomly chosen for the purpose of the plot example):

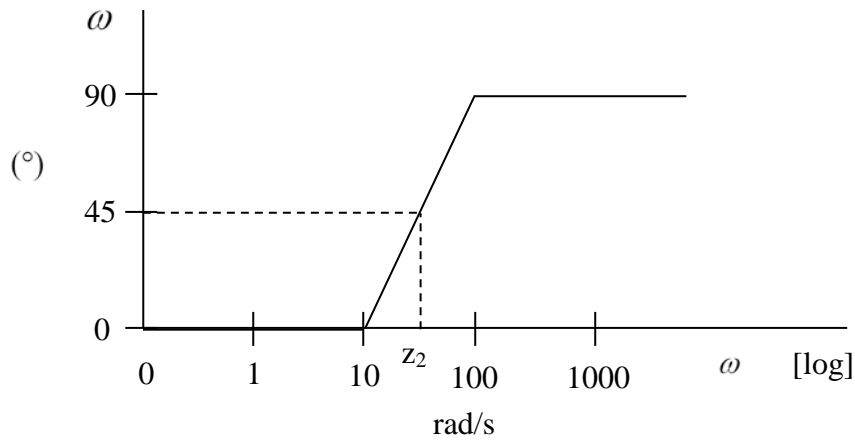
Term (constant): -90°



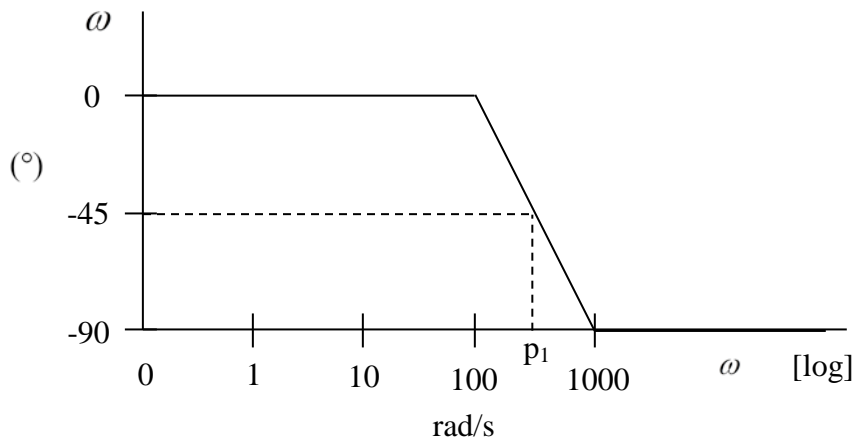
Term (zero): $\tan^{-1}(\omega / z_1)$



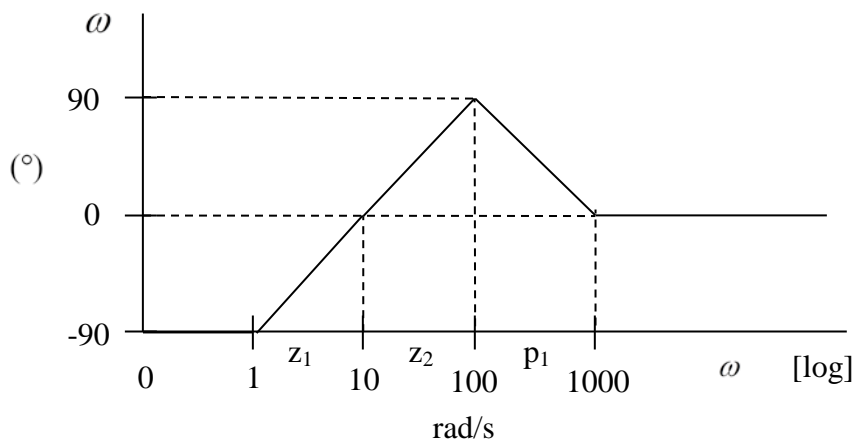
Term (zero): $\tan^{-1}(\omega / z_2)$



Term (pole): $-\tan^{-1}(\omega / p_1)$



Now, using superposition, we can sum the contributions from each of the phase terms (previous 4 plots), yielding the overall phase plot.



To refine your plot, you may smooth the boundaries between poles/zeros for a more realistic plot.

The amplitude and phase plots are closely related: for every up (down) kink in log amplitude there is an up (down) step in phase angle.

IV. Estimation Methods

First, convert your transfer function to standard form, as in Eqn. (5). For each pole and zero, determine where the break frequency ω is, by solving for each respective term. For example:

$$H(j\omega) = \frac{z_1 z_2 (j\omega / z_1 + 1)(j\omega / z_2 + 1)}{j\omega / p_1 (j\omega / p_1 + 1)} \quad (20)$$

Magnitude

Then use these general rules for sketching your magnitude curve:

- 1) Find the initial value of the transfer function as $\omega \rightarrow 0$ and $\omega \rightarrow \infty$. These are the values of your lower and upper frequency bounds, respectively
- 2) For every **pole**, the slope of the magnitude curve **decreases** by 1, where at the break frequency, the curve value is **decreased** -3dB.

For every **zero**, the slope of the magnitude curve **increases** by 1, where at the break frequency, the curve value is **increased** +3dB.

Note: A slope of 1 is equal to 20dB/decade (8dB/octave)

Poles and zeros can also occur at $s = 0$. In the following case, a zero occurs at $s = 0$:

$$H(s) = \frac{s(s / 10 + 1)}{10000(s / 100 + 1)(s / 1000 + 1)} \quad (21)$$

For this example: zeros occur at $s = 0$ and $s = 10$, poles occur at $s = 100$ and $s = 1000$.

Zeros at $s = 0$ increase the initial slope by 1, while poles at $s = 0$ decrease the initial slope by 1. If there are no zeros/poles at $s = 0$, the initial slope of the magnitude curve is 0.

Phase

Then use these general rules for sketching your phase curve:

- 1) For every **pole**, the angle of the phase plot **decreases** by 90° , where at the break frequency, the curve value is **decreased** 45° .

For every **zero**, the angle of the phase plot **increases** by 90° , where at the break frequency, the curve value is **increased** 45° .

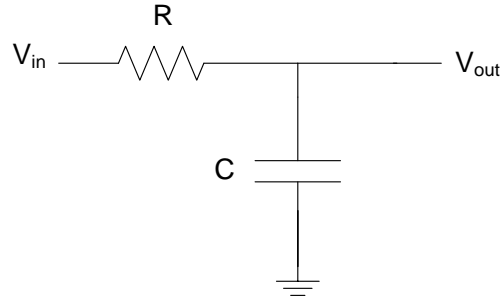
Zeros at $s = 0$ **increase** the initial phase by 90° , while **poles** at $s = 0$ **decrease** the initial slope by 90° . These "steps" are smooth and their transitions are centered around the pole and zero values on the frequency (ω) axis. If there are no zeros/poles at $s = 0$, the initial value of the phase curve is 0° .

Note: You may also use MATLAB or another graphing program to make your Bode plots for homework (but you can't do this on tests, so know how to do them by hand).

V. References

1. F.L. Lewis 2004. EE 4443/4329 - Bode plots lecture notes. The University of Texas at Arlington.
2. R.V. Jones. EE105, Spring 2000 lecture notes. Harvard University.
3. Wikipedia : Bode Plots. http://en.wikipedia.org/wiki/Bode_plot

VI. Example 1: Passive Low Pass Filter



Starting with the transfer function of a passive low pass filter

$$H(j\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC} \quad (22)$$

Magnitude

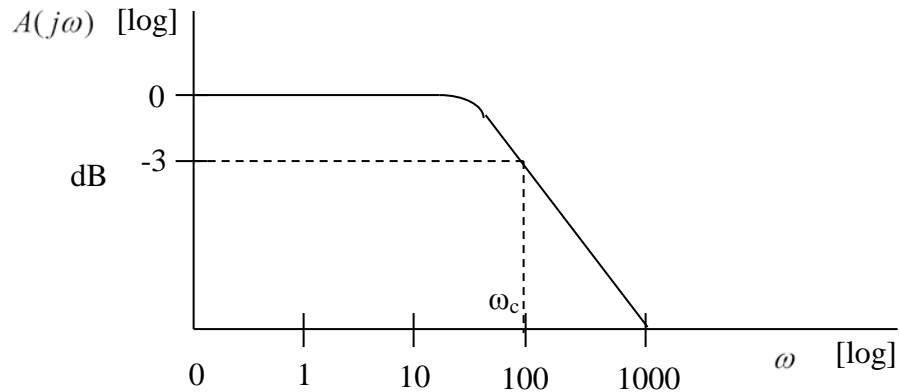
$$|H(j\omega)| = \left| \frac{1}{1 + j\omega RC} \right| = \frac{1}{|1 + j\omega RC|} \quad (23)$$

Using Eqn. (2):

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \quad (24)$$

$$A(j\omega) = 20 \log_{10} |H(j\omega)| = 20 \log_{10} \left(\frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \right) \text{ in units of dB} \quad (25)$$

Sample magnitude plot (cutoff frequency depends on components values R,C):



Looking at the limits of frequency, as $\omega \rightarrow 0$, $A(j\omega) \rightarrow 0$. As $\omega \rightarrow \infty$, $A(j\omega) \rightarrow -\infty$.

Phase

$$\angle H(j\omega) = \angle \frac{1}{1 + j\omega RC} \quad (26)$$

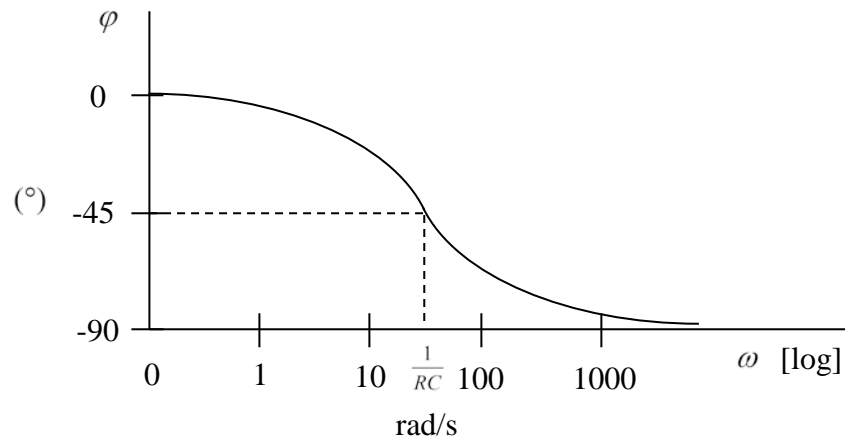
$$\angle H(j\omega) = \angle 1 - \angle(1 + j\omega RC) \quad (27)$$

Since the phase angle of a real constant is 0° :

$$\angle H(j\omega) = -\angle(1 + j\omega RC) \quad (28)$$

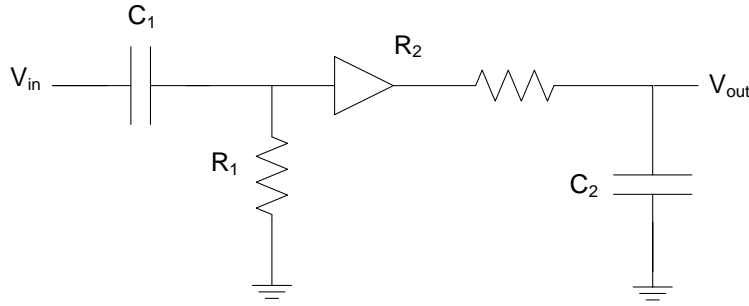
$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega RC}{1}\right) \quad (29)$$

Sample phase plot:



Looking at the limits of frequency, as $\omega \rightarrow 0$, $\angle H(j\omega) \rightarrow 0$. As $\omega \rightarrow \infty$, $\angle H(j\omega) \rightarrow -90$.

VII. Example 2: Bandpass (High Pass – Low Pass) Filter



Starting with the transfer function

$$H(j\omega) = \frac{V_{out}}{V_{in}} = \left(\frac{j\omega R_1 C_1}{1 + j\omega R_1 C_1} \right) \left(\frac{1}{1 + j\omega R_2 C_2} \right) \quad (30)$$

Multiplying the two filter transfer functions together

$$H(j\omega) = \frac{j\omega R_1 C_1}{-\omega^2 R_1 C_1 R_2 C_2 + j\omega R_1 C_1 + j\omega R_2 C_2 + 1} \quad (31)$$

Grouping real and imaginary terms

$$H(j\omega) = \frac{j\omega R_1 C_1}{1 - \omega^2 R_1 C_1 R_2 C_2 + j\omega(R_1 C_1 + R_2 C_2)} \quad (32)$$

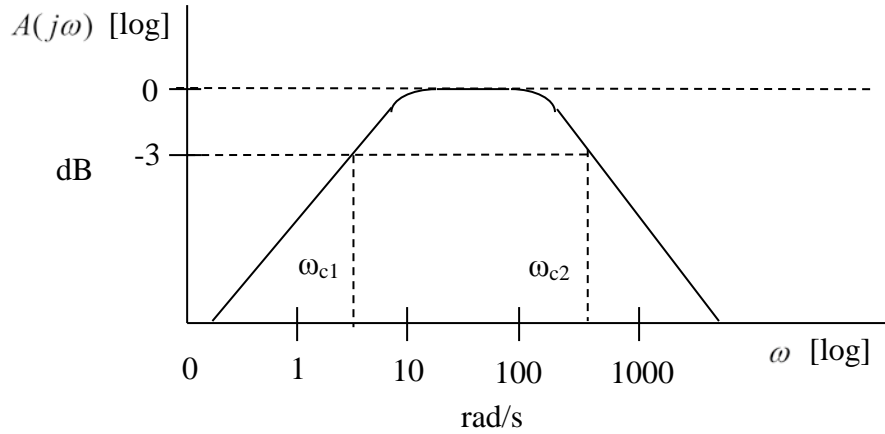
Magnitude

$$|H(j\omega)| = \left| \frac{j\omega R_1 C_1}{1 - \omega^2 R_1 C_1 R_2 C_2 + j\omega(R_1 C_1 + R_2 C_2)} \right| = \frac{|j\omega R_1 C_1|}{|1 - \omega^2 R_1 C_1 R_2 C_2 + j\omega(R_1 C_1 + R_2 C_2)|} \quad (33)$$

$$|H(j\omega)| = \frac{\omega R_1 C_1}{\sqrt{(1 - \omega^2 R_1 C_1 R_2 C_2)^2 + (\omega(R_1 C_1 + R_2 C_2))^2}} \quad (34)$$

$$A(j\omega) = 20 \log_{10} |H(j\omega)| = 20 \log_{10} \left(\frac{\omega R_1 C_1}{\sqrt{(1 - \omega^2 R_1 C_1 R_2 C_2)^2 + (\omega(R_1 C_1 + R_2 C_2))^2}} \right) \quad (35)$$

Sample magnitude plot (shape and cutoff frequencies depend on components values R,C):



Looking at the limits of frequency, as $\omega \rightarrow 0$, $A(j\omega) \rightarrow 0$. As $\omega \rightarrow \infty$, $A(j\omega) \rightarrow 0$.

Phase

$$\angle H(j\omega) = \angle \frac{j\omega R_1 C_1}{1 - \omega^2 R_1 C_1 R_2 C_2 + j\omega(R_1 C_1 + R_2 C_2)} \quad (36)$$

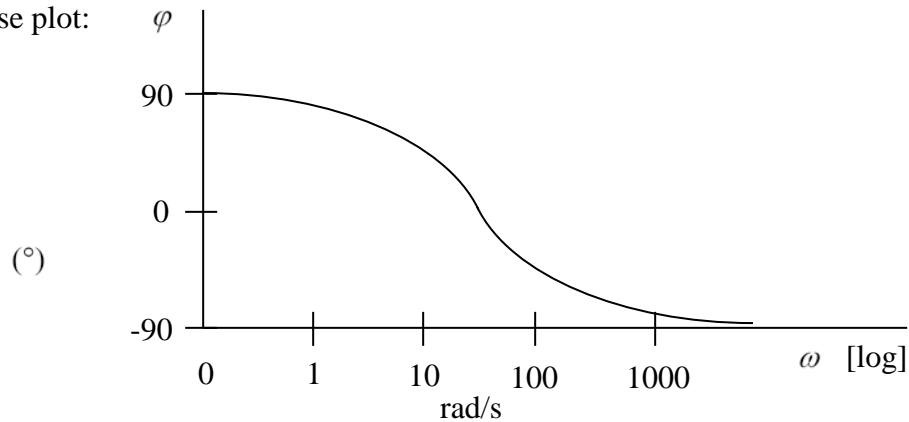
$$\angle H(j\omega) = \angle j\omega R_1 C_1 - \angle(1 - \omega^2 R_1 C_1 R_2 C_2 + j\omega(R_1 C_1 + R_2 C_2)) \quad (37)$$

From Eqn. (12):

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega R_1 C_1}{0}\right) - \tan^{-1}\left(\frac{\omega(R_1 C_1 + R_2 C_2)}{1 - \omega^2 R_1 C_1 R_2 C_2}\right) \quad (38)$$

$$\angle H(j\omega) = 90 - \tan^{-1}\left(\frac{\omega(R_1 C_1 + R_2 C_2)}{1 - \omega^2 R_1 C_1 R_2 C_2}\right) \quad (39)$$

Sample phase plot:



Looking at the limits of frequency, as $\omega \rightarrow 0$, $\angle H(j\omega) \rightarrow 90$. As $\omega \rightarrow \infty$, $\angle H(j\omega) \rightarrow -90$.