

# Green's examples

VALUE-VALUE BC :

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad \text{with} \quad \begin{cases} u(x,0) = g(x) : \text{I.C.} \\ u(0,t) = u_0(t) : \text{B.C. @ 0} \\ u(L,t) = u_L(t) : \text{B.C. @ L} \end{cases}$$

1) Homogeneous with constant I.C. :  $\begin{cases} Q(x,t) \equiv 0 \\ g(x) \equiv 1 \\ u_0(t) \equiv u_L(t) \equiv 0 \end{cases}$

$$u(x,t) = \int_0^L g(x_0) G(x,t; x_0, 0) dx_0$$

where  $G(x,t; x_0, t_0) = \sum_{m=1}^{\infty} \frac{2}{L} \sin\left(\frac{m\pi x_0}{L}\right) \sin\left(\frac{m\pi x}{L}\right) e^{-k\left(\frac{m\pi}{L}\right)^2(t-t_0)}$

$$u(x,t) = \sum_{m=1}^{\infty} \underbrace{\frac{2}{L} \int_0^L \sin\left(\frac{m\pi x_0}{L}\right) dx_0}_{C_m} \sin\left(\frac{m\pi x}{L}\right) e^{-k\left(\frac{m\pi}{L}\right)^2 t}$$

$$C_m = \frac{2}{L} \frac{L}{m\pi} \left[ -\cos\left(\frac{m\pi x_0}{L}\right) \right]_0^L = \frac{2}{m\pi} (1 - \cos(m\pi))^{(-1)^m}$$

$$u(x,t) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{m\pi x}{L}\right) e^{-k\left(\frac{m\pi}{L}\right)^2 t} \quad \text{with } C_m = \begin{cases} \frac{4}{m\pi} & \text{for } m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

2) Inhomogeneous PDE with constant driving:  $\left\{ \begin{array}{l} Q(x,t) \equiv 1 \\ J(x) \equiv 0 \\ u_0(t) \equiv u_L(t) \equiv 0 \end{array} \right.$

$$u(x,t) = \iint_0^L Q(x_0, t_0) G(x, t; x_0, t_0) dx_0 dt_0$$

$$= \sum_{n=1}^{\infty} \underbrace{\left( \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x_0}{L}\right) dx_0 \right)}_{C_n} \cdot \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{for } n \text{ odd}} \cdot \underbrace{\int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)} dt_0}_{\text{otherwise}}$$

$$C_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)} dt_0 = -\frac{1}{k\left(\frac{n\pi}{L}\right)^2} \cdot \int_0^{k\left(\frac{n\pi}{L}\right)^2 t} e^{-\lambda} d\lambda$$

$$\lambda = k\left(\frac{n\pi}{L}\right)^2(t-t_0)$$

$$d\lambda = -k\left(\frac{n\pi}{L}\right)^2 dt_0$$

$$= \frac{1}{k\left(\frac{n\pi}{L}\right)^2} \left[ e^{-\lambda} \right]_0^{k\left(\frac{n\pi}{L}\right)^2 t} = \frac{1}{k\left(\frac{n\pi}{L}\right)^2} \left( 1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} d_n \cdot \sin\left(\frac{n\pi x}{L}\right) \left( 1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right)$$

with  $d_n = \begin{cases} \frac{4}{n\pi} \cdot \frac{1}{k\left(\frac{n\pi}{L}\right)^2} = \frac{4L^2}{k(n\pi)^3} & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$

3) Inhomogeneous B.C. constant in time :  $\left\{ \begin{array}{l} Q(x,t) = 0 \\ g(x) = 0 \\ u_0(t) = T_0 \quad (@x=0) \\ u_L(t) = T_L \quad (@x=L) \end{array} \right.$

$$\begin{aligned}
 u(x,t) &= \int_0^t u_0(t_0) + k \frac{\partial}{\partial x_0} G(x,t; 0, t_0) dt_0 \\
 &\quad - \int_0^t u_L(t_0) + k \frac{\partial}{\partial x_0} G(x,t; L, t_0) dt_0 \\
 &= T_0 + \sum_{n=1}^{\infty} \frac{2}{L} \frac{n\pi}{k} \sin\left(\frac{n\pi x}{L}\right) \int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)} dt_0 \\
 &\quad - T_L + \sum_{n=1}^{\infty} \frac{2}{L} \frac{n\pi}{k} \underbrace{\cos(n\pi t)}_{(-1)^n} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{same}} \int_0^t \dots dt_0 \\
 &= \sum_{n=1}^{\infty} \underbrace{k \cdot \frac{2}{L} \cdot \frac{n\pi}{k} \cdot \frac{1}{k\left(\frac{n\pi}{L}\right)^2}}_{\frac{2}{n\pi}} \left( T_0 - T_L (-1)^n \right) \sin\left(\frac{n\pi x}{L}\right) \left( 1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{e.g.: } T_0 = T_L = T \Rightarrow u(x,t) &= T \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \left( 1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right) \\
 &= T - T \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}
 \end{aligned}$$

$$\text{with } c_n = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

## More Green's examples

FLUX - FLUX BC:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad \text{with} \quad \begin{cases} u(x,0) = g(x) & \text{I.C.} \\ \frac{\partial u}{\partial x}(0,t) = u'_0(t) & \text{B.C.} @ 0 \\ \frac{\partial u}{\partial x}(L,t) = u'_L(t) & \text{B.C.} @ L \end{cases}$$

$$\rightarrow G(x,t; x_0, t_0) = \frac{1}{L} + \sum_{n=1}^{\infty} \frac{2}{L} \cos\left(\frac{n\pi x_0}{L}\right) \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)}$$

1) Homogeneous with constant I.C. :

$$\begin{cases} Q(x,t) \equiv 0 \\ g(x) \equiv 1 \\ u'_0(t) \equiv u'_L(t) \equiv 0 \end{cases}$$

$$u(x,t) = \int_0^L g(x_0) G(x,t; x_0, 0) dx_0$$

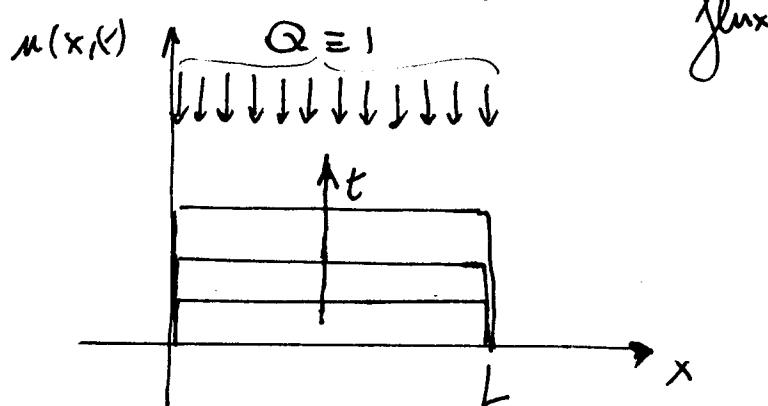
$$= \underbrace{\frac{1}{L} \int_0^L dx_0}_1 + \underbrace{\sum_{n=1}^{\infty} \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x_0}{L}\right) dx_0}_{\frac{L}{n\pi} \left[ \sin\left(\frac{n\pi x_0}{L}\right) \right]_0^L} \cdot \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} = 0 !$$

$u(x,t) \equiv 1$  no change ! (steady state solution with zero flux)

2) Inhomogeneous PDE with constant driving source:  $\left\{ \begin{array}{l} Q(x,t) = 1 \\ g(x) = 0 \\ u'_0(t) = u'_L(t) = 0 \end{array} \right.$

$$\begin{aligned} u(x,t) &= \iint_0^t Q(x_0, t_0) G(x, t; x_0, t_0) dx_0 dt_0 \\ &= \underbrace{\frac{1}{L} \int_0^L dx_0 \int_0^t dt_0}_{\text{constant}} + \sum_{m=1}^{\infty} \underbrace{\frac{2}{L} \int_0^L \cos\left(\frac{m\pi x_0}{L}\right) dx_0}_{\text{harmonic}} \underbrace{\cos\left(\frac{m\pi x}{L}\right)}_{\text{harmonic}} \int_0^t e^{-k\left(\frac{m\pi}{L}\right)^2(t-t_0)} dt_0 \\ &\quad \underbrace{\left[ \sin\left(\frac{m\pi x_0}{L}\right) \right]_0^L}_{} = 0 \end{aligned}$$

$u(x,t) = t$  ! unbounded uniform growth, in absence of flux



3) Inhomogeneous B.C., constant flux over time :  $\begin{cases} Q(x,t) = 0 \\ g(x) = 0 \\ -k\mu_0'(t) = \dot{\Phi}_0 \quad (@x=0) \\ -k\mu_L'(t) = -\dot{\Phi}_L \quad (@x=L) \end{cases}$



$$u(x,t) = - \int_0^t k\mu_0'(t_0) G(x,t; 0, t_0) dt_0 + \int_0^t k\mu_L'(t_0) G(x,t; L, t_0) dt_0$$

$$\begin{aligned} &= \dot{\Phi}_0 \int_0^t G(x,t; 0, t_0) dt_0 + \dot{\Phi}_L \int_0^t G(x,t; L, t_0) dt_0 \\ &= \dot{\Phi}_0 \left( \frac{1}{L} \int_0^t dt_0 + \sum_{n=1}^{\infty} \frac{2}{L} \cos\left(\frac{n\pi x}{L}\right) \underbrace{\int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)} dt_0}_{\frac{L^2}{k(n\pi)^2} (1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t})} \right) \\ &\quad + \dot{\Phi}_L \left( \frac{1}{L} \int_0^t dt_0 + \sum_{n=1}^{\infty} \frac{2}{L} \cos(n\pi) \cos\left(\frac{n\pi x}{L}\right) \underbrace{\int_0^t e^{-k\left(\frac{n\pi}{L}\right)^2(t-t_0)} dt_0}_{(-1)^n \frac{L^2}{k(n\pi)^2} (1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t})} \right) \end{aligned}$$

$$u(x,t) = \frac{\dot{\Phi}_0 + \dot{\Phi}_L}{L} t + \sum_{n=1}^{\infty} \frac{2L}{k(n\pi)^2} \left( \dot{\Phi}_0 + (-1)^n \dot{\Phi}_L \right) \cos\left(\frac{n\pi x}{L}\right) \left( 1 - e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right)$$

$$\lim_{t \rightarrow \infty} : u(x, t \rightarrow \infty) = \frac{\dot{\Phi}_0 + \dot{\Phi}_L}{L} t + \sum_{n=1}^{\infty} \frac{2L}{k(n\pi)^2} \left( \dot{\Phi}_0 + (-1)^n \dot{\Phi}_L \right) \cos\left(\frac{n\pi x}{L}\right)$$