Modeling the BP Oil Spill of 2010:
A Simplified Model of Oil Diffusion in Water

BENG 221 – Fall 2012

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Introduction

On April 20, 2010, the *Deepwater Horizon* mobile offshore drilling unit owned by British Petroleum (BP) experienced a catastrophic explosion that caused it to collapse into the bottom of the Gulf of Mexico (Figure 1). In the process, a sea-floor oil gusher that was connected to the unit ruptured and started to spill oil into the Gulf. The spill lasted for about 5 months unabated and was finally sealed off on September 19, 2010. However, by that time, the damage was already done. The oil spill, known by many as the BP Oil Spill, had a huge impact on the Gulf of Mexico and the environment. It polluted marine and wildlife habitats; it mutated several marine life forms; it damaged the fishing and tourism industry; and it resulted in the loss of 4.9 million barrels of crude oil. President Barack Obama described this incident as the "worst environmental disaster America has ever faced."

Figure 1: The explosion that triggered the BP Oil Spill. Image taken from Inhabitat.com

Problem Setup

Despite this tragic incidence, there is opportunity present for those in academia. The oil spill can be modeled as the diffusion of oil in water, and the concentration of oil over time can be determined based on the diffusivity of oil in water. However, in order to model the diffusion of oil in water, it is important to remember that oil does not just diffuse in water. There are many reactions and external forces at work that can alter the distribution and concentration of oil in water, such as biodegradation, sedimentation, dissolution, dispersion, emulsification, evaporation, and oxidation (Figure 2). In addition, the water currents in the Gulf of Mexico can act as a driving force and cause the oil to spread out faster throughout the Gulf.
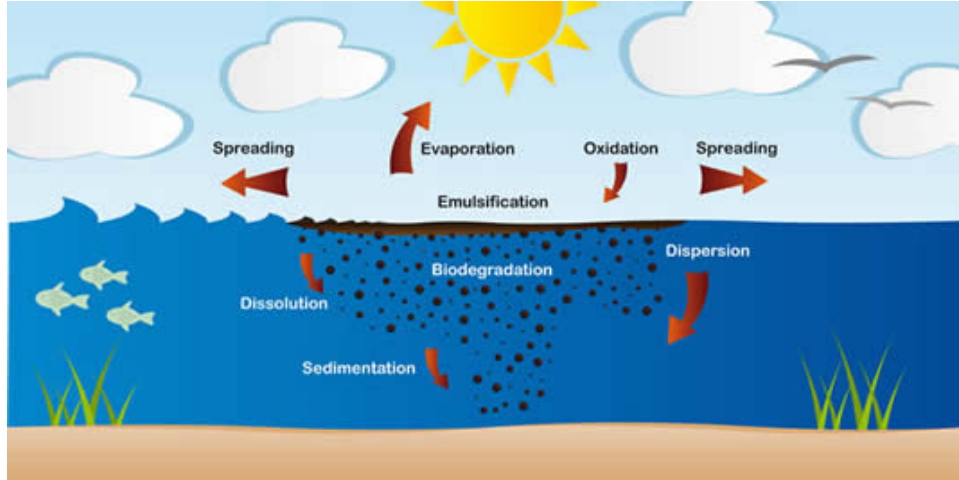
Figure 2: The Fate of an Oil Spill. Image taken from ITOPF.com.

Therefore, to accurately measure the concentration of oil over time, it is important to take into consideration all of the above reactions and forces into the model. However, these factors can complicate the model, making it difficult to find a suitable solution.

To simplify the model so that an analytical and numerical solution can be achieved, the following assumptions were made:
1. No boundary on oil spill, as size of ocean is much greater than scale of oil spill.
2. No biodegradation, sedimentation, dissolution, dispersion, emulsification, evaporation, oxidation, and driving forces.
3. Constant oil source at a specific point in space.
4. Instant rise of oil to the surface of the Gulf, thereby making the z-axis irrelevant

These assumptions simplify the model such that the only force examined is the diffusion of oil in water. Given these assumptions, the partial differential equation that describes the model is simplified to the following:

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

The initial and boundary conditions, respectively, are as follows:

$$u(x,y,t=0) = \delta(x)\delta(y)$$
$$\begin{cases} u(x=\infty,y,t) = 0 \\ u(x,y=\infty,t) = 0 \end{cases}$$

where the $\delta(x)$ and $\delta(y)$ functions imply infinite concentration at the point source, meaning that at $(x,y) = (0,0)$, the concentration of oil is at infinity (i.e., no dilution with water), while at $(x,y) \neq (0,0)$, the concentration of oil is zero.

Two methods were used to solve this model. For the analytical solution, the Green's function for infinite domain was used; for the numerical solution, the pdepe function from MATLAB was used.

Analytical Solution

To solve the model analytically, first solve for the Green's Function. Start by finding the 1D Green's function solution and then extrapolate to 2D.

1. Write out 1D diffusion equation with IC and BC. The driving term is a delta function.

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad \begin{cases} u(\pm\infty,t)=0 \\ u(x,t=0)=\delta(x) \end{cases}$$

$$*D = 1.13\times10^{-3}\,\frac{m^2}{day}$$

*See Appendix for Derivation of Diffusion Coefficient, D

2. Perform Fourier Transform in the Spatial Domain since the x domain is infinite.

$$U(x,t) \rightarrow U_F(\omega,t)$$

$$\frac{\partial u_F}{\partial t} = -D\omega^2 U_F$$

3. Perform Laplace Transform in the Time Domain since t is defined for all time $t>0$. The Laplace transform of the delta function is 1.

$$U_F(\omega,t) \rightarrow U_{FL}(\omega,s)$$

$$sU_{FL} - U_F(\omega,0) = -D\omega^2 U_{FL}$$

$$U_F(\omega,0) = L\{\delta\} = 1$$

4. Solve the resulting equation for U_{FL} algebraically

$$U_{FL} = \frac{1}{s+D\omega^2}$$

5. Perform Inverse Transforms to get back u.

$$L^{-1}\{U_{FL}\} = U_F = e^{-D\omega^2}$$

$$F^{-1}\{U_F\} = u = \frac{1}{\sqrt{4\pi Dt}}e^{\frac{-x^2}{4Dt}}$$

6. Extrapolate to 2D. With the addition of the y coordinate, we would have an additional Fourier Transform and inverse transform. This would change our Green's function by an additional factor of $\dfrac{1}{\sqrt{4\pi D(t-t_0)}}$.

$$G\left(x,t,x_0,t_0\right) = \frac{1}{4\pi D(t-t_0)} e^{\frac{-[(x-x_0)^2+(y-y_0)^2]}{4D(t-t_0)}}$$

The general solution using the Green's Function is as follows:

$$u(x,t) = \int_0^L g(x_0)G(x,t;x_0,0)dx_0 + \int_0^L \int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)dx_0\,dt_0$$

$$+ \int_0^t u_0(t_0)D\frac{\partial}{\partial x_0}G(x,t;0,t_0)dt_0 - \int_0^t u_L(t_0)G(x,t;L,t_0)dt_0$$

Since the model does not have a driving force, and the concentration of oil at the boundaries is zero, the solution simplifies to just the following:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G\left(x,t,x_0,t_0\right)\delta(x_0)\delta(y_0)dx_0\,dy_0\,dt_0$$

$$= \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{4\pi D(t-t_0)} e^{\frac{-[(x-x_0)^2+(y-y_0)^2]}{4D(t-t_0)}} dx_0\,dy_0\,dt_0$$

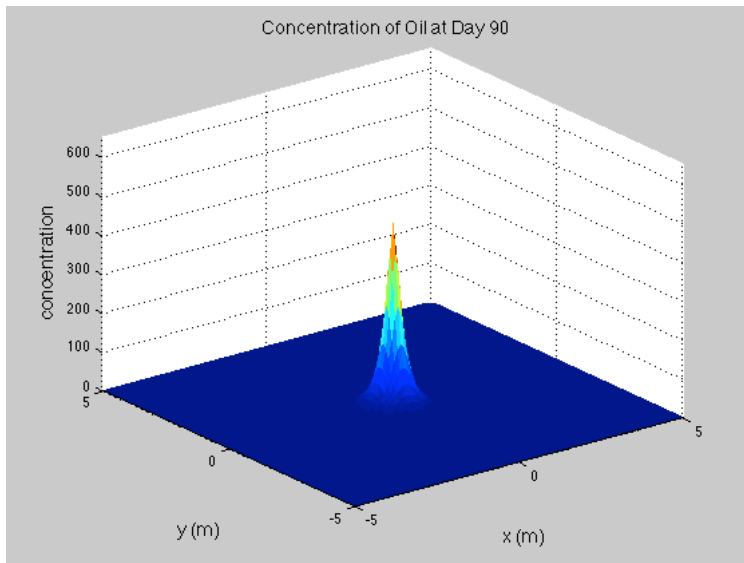$$= \int_0^t \frac{1}{4\pi D(t-t_0)} e^{\frac{-(x^2+y^2)}{4D(t-t_0)}} dt_0$$

The integral above cannot be computed directly, so the answer was approximated using the Midpoint Approximation Method. Using this method, the concentration of oil can be plotted over an interval of 150 days (or about 5 months, the duration of the BP oil spill).
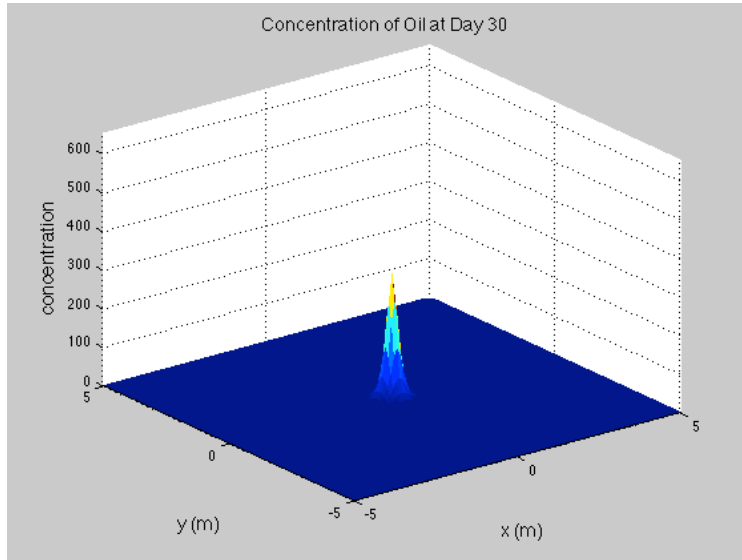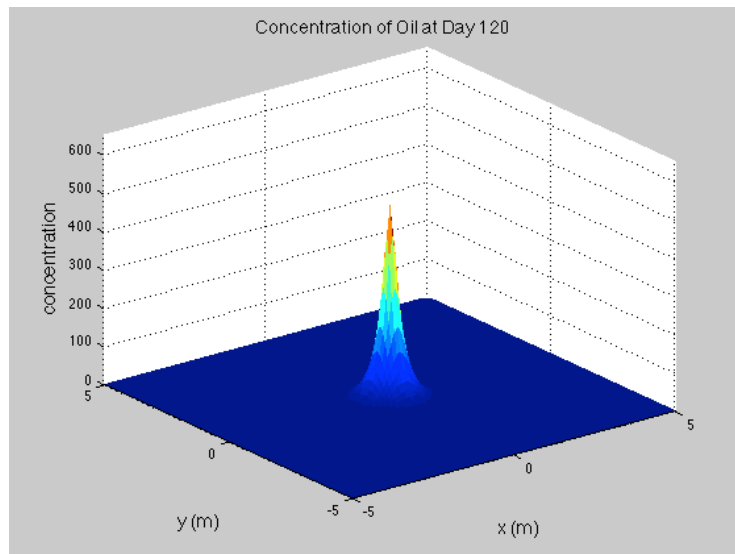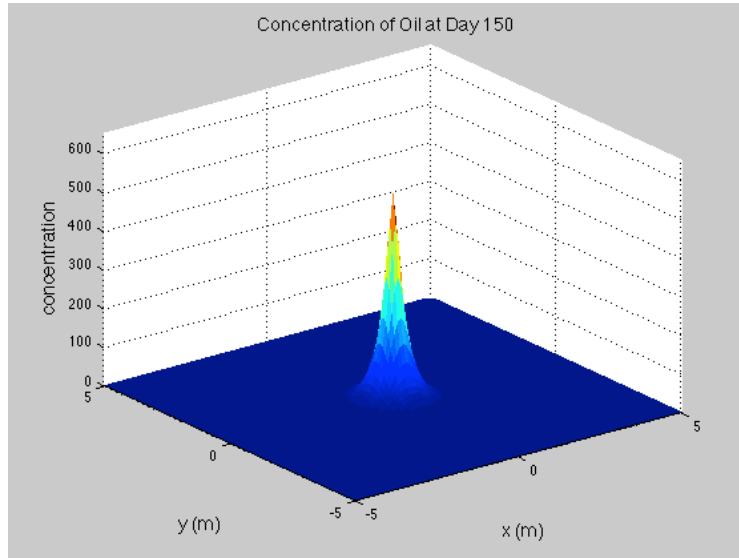
Figure 3: Concentration of Oil at Day 30
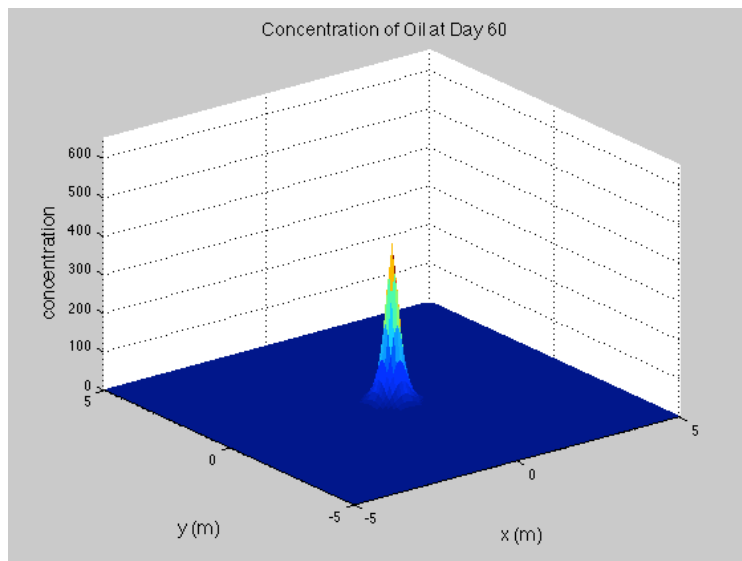
Figure 4: Concentration of Oil at Day 60

Figure 5: Concentration of Oil at Day 90

Figure 6: Concentration of Oil at Day 120

Figure 7: Concentration of Oil at Day 150

It is important to note that the initial condition is set to an infinite concentration that transcends the bounds of the y-axis in the graphs. This initial condition is such that the concentration of oil is infinity when $(x,y)=(0,0)$ and zero when $(x,y)\neq(0,0)$. Because this initial condition takes place at only one point in the x-y domain, it results in an extremely thin vertical line that is impossible to distinguish at the current resolution. It is important to take this 'invisible' vertical line into account when analyzing the graphs. Figures 3 to 7 show the progression of the diffusion of oil through time and space. As time goes by, oil slowly spreads radially. The concentration profile maintains a Gaussian distribution, suggesting that most oil remains at the site of the spill. As the oil diffuses, the concentration decreases. Figure 7 shows that this Gaussian distribution is wider at the bottom after 150 days. However, there is no change at the origin, suggesting that oil is still mainly concentrated at the source. After 150 days, the spread is only diffused within a 2m^2 radius.

Based on the magnitude of the damage and the controversy arising from the BP oil spill, the results of the analytical solution seem surprising. However, this problem was solved under various assumptions that make the spread of oil only dependent on diffusion. The small magnitude of the diffusion coefficient suggests that diffusivity plays a minimal role on the dispersion of oil, and in order to realistically simulate an oil spill, other forces must be taken into account.
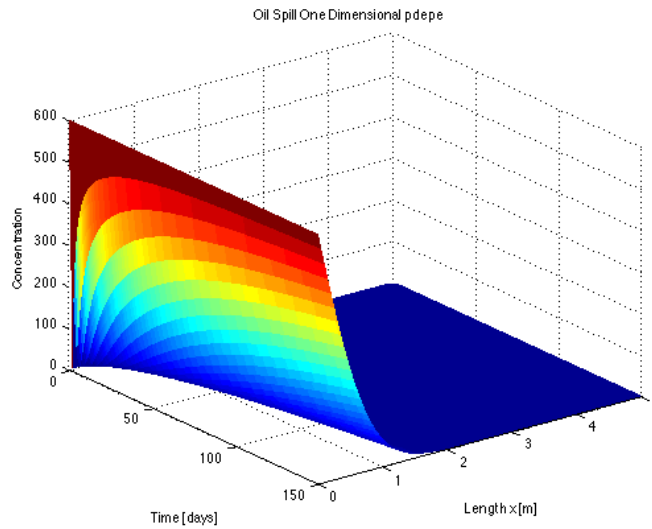
Numerical Solution
A similar solution can be obtained, using MATLAB's pdepe function (see appendix for code). For the numerical solution, a one-dimensional approach was used to simplify the problem, as it is symmetrical from the source of the oil spill to all direction in the plane. Based on the analytical solution we limited the x-axis to 5 meters, as further simulation would not make a difference.

The initial conditions of this model consisted of a delta function in the x domain, meaning the concentration of oil will be infinite at the origin and zero elsewhere at time $t=0$. To simulate this initial condition in our pdepe function, a relatively large value (600) was used as our initial condition. However, because this is a one-dimensional solution, a left value boundary condition

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was applied at the initial concentration of the spill to simulate the delta function. Assuming infinite concentration at the origin of the spill, it can be generalized that a high concentration of oil at that boundary will be constantly maintained throughout time.

Figure 8: One-Dimensional Oil Spill, Using MATLAB's pdepe Function

The numerical solution depicted by Figure 8, shows a similar trend seen in the analytical solution. At $t = 0$ and $x = 0$, the oil spill is at a maximum concentration. As time progresses, the concentration spreads out towards the x direction. Based on Figure 8, it can be observed that diffusion is extremely slow, because after 150 days, the majority of the oil still remains at the origin, with some oil diffusing within a 2m^2 area. This is also depicted by the concentration profile at different time points as observed in Figure 9.
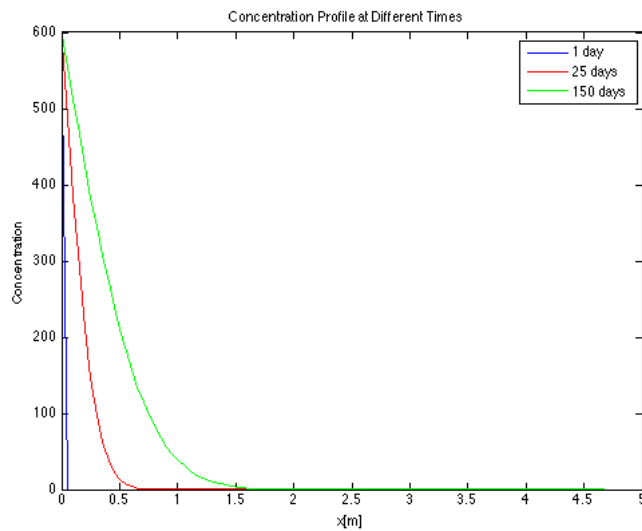
Figure 9: Concentration Profile at Different Times

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Discrepancies between the Analytical and Numerical Solution

The main discrepancies between these two methods should arise from the simulation of the initial condition. In the analytical solution, the initial condition is set such that the source at the origin has an infinite concentration. However, for the numerical solution, a relatively large number (600) was utilized to emulate an infinitely high concentration and used as a left value boundary condition. Furthermore, instead of an infinite boundary, the numerical solution was simulated to have a boundary at x=5, as simulation beyond that point would make no difference. Judging from the graphical solutions obtained by both methods, the discrepancies resulting from this adaptation on the initial conditions do not affect the solutions. Both methods exhibit a similar trend: the maintenance of high concentration at the origin and the slow diffusion of oil after 150 days. Both solutions show that after 150 days, the oil is still constrained within a 2m^2 region, following a Gaussian distribution from the origin of the spill.

Conclusion and Future Works

Both the analytical and numerical solutions show that the diffusion of oil follows a Gaussian trend, with the oil remaining primarily at the source of the spill and spreading within a 2m^2 area after 150 days. However, this simplified model contradicts the real magnitude of the BP Oil Spill, demonstrating that diffusion alone is not enough to promote the spreading of oil in water. Other forces and reactions need to be taken into account to obtain a real-life model of the oil spill. Future directions should include a term that corresponds to the Gulf's water current, a term that corresponds to the reactions that occur between oil and water such as biodegradation, emulsification, and sedimentation, and a sink term that would correspond to the cleanup of oil. The inclusion of these factors would produce a much more realistic model of an oil spill, which can be helpful in predicting the magnitude of an oil spill and facilitating an adequate plan of action.

Appendix

Derivation of Diffusion Coefficient of Oil in Water

In Hamam's 1987 paper, *The Diffusion on Oil in Water*, the diffusion coefficient of oil in water D, in units of cm^2 per hour, was provided as a function of temperature T in Celsius: $D = 4.13 \times 10^{-3} T^{1.53}$. According to data present on the US National Oceanographic Data Center website, the average water temperature of the Gulf of Mexico was calculated to be 22°C. Using this value, the diffusion coefficient of oil in water was calculated to be $D = 0.47 \dfrac{cm^2}{hr}$. The diffusion coefficient was then converted to units of m^2 per day, resulting in the final value of

$$D = 1.13 \times 10^{-3} \frac{m^2}{day}.$$

Code for Finding and Plotting Analytical Solution

```
%select the time at which to see solution
t=5*30;

%initiate
D=(0.5*10^-4)*24; %[m^2/day]
L=5; %how far are we looking in [m]
dx=.1;
xmesh= -L:dx:L;
nx= length(xmesh);
dy=.1;
ymesh= -L:dx:L;
ny= length(ymesh);
unum=zeros(nx,ny);

%set up integration
a=0;
b=t;
n=150; %number of steps
dh=5*t/n; %stepwidth
hmesh=0:dh:t; %vector of step locations
nh=length(hmesh);
w=2*dh;

%integrate and plot
for i=2:nh
    for x=1:nx
        for y=1:ny
        unum(x,y)=unum(x,y)+...
            w*exp(-(xmesh(x)^2+ymesh(y)^2)/...
            (4*D*hmesh(i)))/(4*pi*D*hmesh(i));
        end
    end
    figure(1);
    filename = 'integrator.gif';
    surf(xmesh,ymesh,unum,'LineStyle','none')
    % Create xlabel
    xlabel('x (m)','FontSize',14);
```

```matlab
    xlim([-5 5]);
    % Create ylabel
    ylabel('y (m)','FontSize',14);
    ylim([-5 5]);
    zlabel('concentration','FontSize',14);
    zlim([0 650]);
    % Create title
    title(['Concentration of Oil at Day ',num2str(hmesh(i))],'FontSize',14);
    drawnow
    frame = getframe(1);
    im = frame2im(frame);
    [imind,cm] = rgb2ind(im,256);
    if i == 2;
        imwrite(imind,cm,filename,'gif', 'Loopcount',inf);
    else
        imwrite(imind,cm,filename,'gif','WriteMode','append');
    end
end
```

Code for MATLAB's pdepe Plot and Concentration Profile

```matlab
function project_numerical
clc;clear all;close all;
global D c0 L ;

D = 1.3e-3; %m^2/s;
c0= 600; %mmol/m^3
L=5; %m

t_max= 150;

t=linspace(0, t_max, 200);
x=linspace(0,L, 100);
theta=(linspace(0, 2*pi,200));

sol_pdepe = pdepe(0,@pdefun,@ic,@bc,x,t);

sol_pdepe_t=sol_pdepe';

figure(1)
surf(t,x,sol_pdepe_t, 'EdgeColor', 'none')
title('Oil Spill One Dimensional pdepe')
xlabel('Time [days]')
ylabel('Length x [m]')
zlabel('Concentration ')

figure(2)
h1=plot(x, sol_pdepe_t(:,1),'b');
hold on
%plot(x, sol_pdepe_t(:,50));
h2=plot(x, sol_pdepe_t(:,25), 'r');
%plot(x, sol_pdepe_t(:,100));
h3=plot(x, sol_pdepe_t(:,150), 'g');
```

```matlab
legend([h1, h2, h3],{'1 day','25 days', '150 days'});
title(['Concentration Profile at Different Times']);
xlabel('x[m]');
ylabel('Concentration');

end


% function definitions for pdepe:
% -----------------------------------------------------------

function [c, f, s] = pdefun(x, t, u, DuDx)
% PDE coefficients functions

global D
c = 1;
f = D * DuDx; % diffusion
s = 0; % homogeneous, no driving term
end


% -----------------------------------------------------------

function u0 = ic(x)
% Initial conditions function
u0=(x==0);
end


% -----------------------------------------------------------

function [pl, ql, pr, qr] = bc(xl, ul, xr, ur, t)
% Boundary conditions function

global c0

pl = ul-600; % c0 value left, chose a large number to simulate infinity
ql = 0;  % arbitrary flux left boundary condition
pr = 0; % zero value right boundary condition
qr = 1;  % no flux right boundary condition

end
```

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