BENG 122A Fall 2025 HW #1

Due *Friday, October 10* at 11:59pm

1. [35 pts] Consider the following ODEs describing the dynamics in the state variables $u(t)$ and $v(t)$ of a biosystem driven by an input $f(t)$:

$$\begin{aligned} \frac{du}{dt} &= v(t) \\ \frac{dv}{dt} &= -a\,v(t) - b\,\sinh(c\,u(t)) + d\,f(t) \end{aligned}$$

with parameters $a = 1\ \text{s}^{-1}$, $b = 1\ \text{m s}^{-2}$, $c = 2\ \text{m}^{-1}$, and $d = 2\ \text{kg}^{-1}$.

   (a) [5 pts] Is this system linear or nonlinear? Explain.

   (b) [10 pts] Describe a physical instance in biology or bioengineering that implements this biosystem. Interpret the variables u, v, and f, and the parameters a through d.

   For a steady-state input $\overline{f} = 1$ N, find the steady-state values $\overline{u}$ and $\overline{v}$ reached in the state variables. *Hint*: Note the inverse for sinh is given by $\sinh^{-1}(x) = \ln(x + \sqrt{1 + x^2})$.

   (c) [10 pts] Formulate the ODEs for the linearized system describing the small-signal dynamics in the state variables $\tilde{u}$ and $\tilde{v}$ around steady-state, for a small perturbation $\tilde{f}$ around steady-state.

2. [20 pts] The cardiovascular system performs many functions that are (literally) vital to our health. Among those are the supply of oxygen and nutrients to the organs across the body, maintenance of balances in metabolism, and regulation of immune response against invading pathogens and malignancies. Here we will just briefly cover two of these, using simple linear compartment models. To start, consider a model of the heart pumping blood through the vasculature, resulting in blood flow Q (in units L/s), and blood pressure P (in units Pa).

   (a) [5 pts] Due to viscosity of the blood plasma, blood flow through the cardiovascular system is subject to resistance, just like in an electrical circuit. Define this resistance R in terms of Q and P, and express its units. Explain how blood pressure rises when plaque forms in the vasculature, narrowing its effective diameter.

   (b) [5 pts] Due to elasticity of the vessel walls and organs as "reservoirs" in the path of the vasculature, total blood volume in the cardiovascular system is subject to compliance C, which is the mechanical equivalent of capacitance in an electrical circuit. Define this compliance C, and express its units.

   (c) [5 pts] Now consider the effect of the combination of resistance and compliance on the first-order dynamics of blood flow and pressure in the cardiovascular system. Define the time constant of this simplified model of the system. Draw an equivalent electrical circuit for a compartment model of the vasculature that accounts for this resistance and compliance.

   (d) [5 pts] Now extend this model to account for balance in potential and kinetic hydraulic energy. Consider how the pressure varies across the vasculature due to gravity under influence of differences in height, and due to (square) velocity under influence of variations in diameter.

3. [45 pts] Now we consider another important function of the cardiovascular system in concert with the pancreas: regulation of glucose metabolism through insulin secretion. Upon a spike in glucose level

after a high-caloric meal, the islet cells in the healthy pancreas release insulin in the bloodstream to facilitate the uptake of the excess glucose by muscle, fat, and liver cells as energy reservoirs for future use. Here we simplify the dynamics of glucose regulation by insulin release as a cascade of two processes: 1) the blood carrying the insulin to these energy reservoirs in in the body; and 2) the kinetics in the reaction catalyzed by insulin, converting glucose to glycogen as energy store.

(a) [10 pts] First we model the uptake of insulin in the bloodstream by considering conservation of mass, under a constant flow of blood Q (in L/s) in and out of the vasculature. We assume that insulin enters the blood by the pancreas at a rate $I(t)$ (in mmol/s), and exits on the other end. We also assume well-mixed conditions across the vasculature such that concentration of insulin C (in mol/L) is uniform across the total blood volume V (in L) at all times. Show that the time evolution of the concentration $C(t)$ is described by the following ODE:

$$\frac{dC}{dt} = \alpha\, I(t) - \frac{1}{\tau}\, C(t). \tag{1}$$

and express the constants α and τ in terms of Q and V.

(b) [10 pts] The kinetics in the conversion from glucose to glycogen can be described by the following rate equation

$$\frac{dG}{dt} = -k\, C(t)\, G(t) \tag{2}$$

where $G(t)$ is the glucose concentration, and k is the reaction rate constant. Write the linearized ODE around the operating point $C \approx C_0$ and $G \approx G_0$. Under what conditions is this linear approximation valid?

(c) [15 pts] Transform the set of linear time-invariant ODEs given by (1) and the linearized (2), with initial conditions $C(0) = C_0$ and $G(0) = G_0$, in the Laplace domain, to arrive at an algebraic relation between glucose concentration $G(s)$ and insulin rate $I(s)$.

(d) [10 pts] Using inverse Laplace transforms, solve for $C(t)$ and $G(t)$ under an impulse activation in insulin, $I(t) = 10\ \mu$mol $\delta(t)$, where $\delta(t)$ is the Delta-Dirac function. Assume $Q = 5$ L/min, $V = 5$ L, $k = 1$ L/s mmol, $G_0 = 1$ mmol/L, and $C_0 = 0$. Sketch the waveforms of your solution for $C(t)$ and $G(t)$.

(e) **BONUS** [extra 10 pts] Solve the full nonlinear set of ODEs (1) and (2), and compare the waveforms for $C(t)$ and $G(t)$ with the approximate linearized solution in (d).