Here we again iterate on two of the problems in the previous homework (HW #1 and #2), now studying the stability of system response, and closing the loop with feedback. As before, you may find it helpful to use the available MATLAB Simulink tools.

1. [45 pts] We now consider stability of the second-order biosystem of Problem 1 in HW #1 and #2, driven by input \( f(t) \), and with output \( u(t) \):

\[
\begin{align*}
\frac{du}{dt} &= v(t) \\
\frac{dv}{dt} &= -a v(t) - b \sinh(c u(t)) + d f(t)
\end{align*}
\]

(a) [20 pts] Prove that this system is stable for any positive values of the system parameters \( a, b, c, \) and \( d \). *Hint:* show that the system response is bounded for any small-signal perturbation in the input around any operating point. In particular, analyze the poles of the linearized system around any operating point \( \bar{f} \) in the input, \( f(t) = \bar{f} + \tilde{f}(t) \).

(b) [25 pts] Now consider the dynamics of the system without any damping, when \( a = 0 \). Show that the system is critically stable, with purely imaginary poles. Plot the output \( u(t) \) for zero input \( f(t) = 0 \), and for \( a = 0, b = 1 \text{ m/s}^{-2}, c = 10 \text{ m}^{-1}, \) and \( d = 10 \text{ kg}^{-1} \). Try different initial conditions in the state variables (initial position \( u(0) \) and initial velocity \( v(0) \)). What do you observe, and why?

2. [55 pts] We now return to the model of regulation of glucose metabolism through insulin secretion, as studied in HW #1 (Problem 3) and HW #2 (Problem 2). Specifically, we consider a simple feedback control strategy, where the insulin secretion \( I(t) \) is directly proportional to the difference between the actual glucose concentration \( G(t) \), and a constant target value \( T \) to maintain a healthy metabolic state:

\[
I(t) = K (G(t) - T) \tag{1}
\]

where \( K \) is the feedback strength as a control parameter in the closed-loop system. As before, insulin is well-mixed in the blood volume \( V \) and decays due to exiting the blood stream at flow rate \( Q \):

\[
\frac{dC}{dt} = \alpha I(t) - \frac{1}{\tau} C(t) \tag{2}
\]

where \( \alpha = 1/V \) and \( \tau = V/Q \). We further consider a source of glucose \( J(t) \) entering the bloodstream, as input from the digestive system. Unlike insulin, glucose is not excreted by the kidneys so it recirculates in the vascular system without decay:

\[
\frac{dG}{dt} = \alpha J(t) - k C(t) G(t) \tag{3}
\]

where the last term, as before, models the kinetics in glycogenesis (conversion of glucose to glycogen) catalyzed by insulin.

\[\text{We ignore the fact that } I(t) \text{ cannot possibly go negative, for now– we’ll deal with this later!}\]
(a) [10 pts] Sketch (or print) an equivalent block diagram of the closed-loop system (1)-(3).

(b) [10 pts] Find the steady-state glucose concentration $G$ for the closed-loop system in response to a steady-state glucose input $J$. Show that, as the feedback strength $K$ approaches infinity, the steady-state glucose concentration approaches the control target $T$.

(c) [20 pts] Analyze the stability of the closed-loop system by linearizing its dynamics for small-signal variations around the steady-state operating point in the glucose input $J(t) \approx J + \bar{J}(t)$, insulin concentration $C(t) \approx \bar{C} + \bar{C}(t)$, and glucose concentration $G(t) \approx \bar{G} + \bar{G}(t)$. Specifically, consider the linearized system response $\bar{G}(t)$ to small perturbations $\bar{J}(t)$. Is the closed-loop system stable? How does the control parameter $K$ affect the dynamics of the system response?

$Hint$: Show that the nonlinear ODE (3) transforms into the following LTI form:

$$\frac{d\bar{G}}{dt} = \alpha \bar{J}(t) - k\bar{G} \bar{C}(t) - k\bar{C} \bar{G}(t)$$

and use it along with the small-signal versions of (1) and (2) to find the closed-loop transfer function $G(s) / J(s)$ and its poles.

(d) [15 pts] Evaluate the response of the closed-loop system (1)-(3) to an initial 8 mmol of glucose in the 4 L blood volume, for a target concentration $T = 1$ mmol/L, and with zero initial insulin in the bloodstream. As before, $\alpha = 0.25$ L$^{-1}$, $\tau = 0.8$ min, and $k = 0.2$ L/s mmol. Show the dynamics in the concentrations of insulin $C(t)$ and glucose $G(t)$, for two values of the feedback strength:

i. $K = 0.01$ L/min; and

ii. $K = 0.1$ L/min.

Explain what you observe.