**BENG 122A Fall 2025 HW #3**

Due *Friday, October 24* at 11:59pm

Here we again iterate on two of the problems in the previous homework (HW #1 and #2), now studying the stability of system response, and closing the loop with feedback. As before, you may find it helpful to use the available MATLAB Simulink tools.

1. [45 pts] We now consider stability of the second-order biosystem of Problem 1 in HW #1 and #2, driven by input $f(t)$, and with output $u(t)$:

$$\frac{du}{dt} = v(t)$$
$$\frac{dv}{dt} = -a\,v(t) - b\,\sinh(c\,u(t)) + d\,f(t)$$

   (a) [20 pts] Prove that this system is stable for any positive values of the system parameters $a$, $b$, $c$, and $d$. *Hint*: show that the system response is bounded for any small-signal perturbation in the input around any operating point. In particular, analyze the poles of the linearized system around any operating point $\overline{f}$ in the input, $f(t) = \overline{f} + \tilde{f}(t)$.

   (b) [25 pts] Now consider the dynamics of the system without any damping, when $a = 0$. Show that the system is critically stable, with purely imaginary poles. Plot the output $u(t)$ for zero input $f(t) = 0$, and for $a = 0$, $b = 1$ m s$^{-2}$, $c = 2$ m$^{-1}$, and $d = 2$ kg$^{-1}$. Try different initial conditions in the state variables (initial position $u(0)$ and initial velocity $v(0)$). What do you observe, and why?

2. [55 pts] We now return to the model of regulation of glucose metabolism through insulin secretion, as studied in HW #1 (Problem 3) and HW #2 (Problem 2). Specifically, we consider a simple feedback control strategy, where the insulin secretion $I(t)$ is directly proportional to the difference[1] between the actual glucose concentration $G(t)$, and a constant target value $T$ to maintain a healthy metabolic state:

$$I(t) = K\,(G(t) - T) \tag{1}$$

where $K$ is the feedback strength as a control parameter in the closed-loop system. As before, insulin is well-mixed in the blood volume $V$ and decays due to exiting the blood stream at flow rate $Q$:

$$\frac{dC}{dt} = \alpha\,I(t) - \frac{1}{\tau}\,C(t) \tag{2}$$

where $\alpha = 1/V$ and $\tau = V/Q$. We further consider a source of glucose $J(t)$ entering the bloodstream, as input from the digestive system. Unlike insulin, glucose is not excreted by the kidneys so it recirculates in the vascular system without decay:

$$\frac{dG}{dt} = \alpha\,J(t) - k\,C(t)\,G(t) \tag{3}$$

where the last term, as before, models the kinetics in glycogenesis (conversion of glucose to glycogen) catalyzed by insulin.

[1] We ignore the fact that $I(t)$ cannot possibly go negative, for now– we'll deal with this later!

(a) [10 pts] Sketch (or print) an equivalent block diagram of the closed-loop system (1)-(3).

(b) [10 pts] Find the steady-state glucose concentration $\overline{G}$ for the closed-loop system in response to a steady-state glucose input $\overline{J}$. Show that, as the feedback strength $K$ approaches infinity, the steady-state glucose concentration approaches the control target $T$.

(c) [20 pts] Analyze the stability of the closed-loop system by linearizing its dynamics for small-signal variations around the steady-state operating point in the glucose input $J(t) \approx \overline{J} + \tilde{J}(t)$, insulin concentration $C(t) \approx \overline{C} + \tilde{C}(t)$, and glucose concentration $G(t) \approx \overline{G} + \tilde{G}(t)$. Specifically, consider the linearized system response $\tilde{G}(t)$ to small perturbations $\tilde{J}(t)$. Is the closed-loop system stable? How does the control parameter $K$ affect the dynamics of the system response?

*Hint*: Show that the nonlinear ODE (3) transforms into the following LTI form:

$$\frac{d\tilde{G}}{dt} = \alpha\, \tilde{J}(t) - k\overline{G}\, \tilde{C}(t) - k\overline{C}\, \tilde{G}(t) \tag{4}$$

and use it along with the small-signal versions of (1) and (2) to find the closed-loop transfer function $\tilde{G}(s) \,/\, \tilde{J}(s)$ and its poles.

(d) [15 pts] Evaluate the response of the closed-loop system (1)-(3) to an initial 10 mmol of glucose in the 5 L blood volume, for a target concentration $T = 1$ mmol/L, and with zero initial insulin in the bloodstream. As before, $\alpha = 0.2$ L$^{-1}$, $\tau = 1$ min, and $k = 1$ L/s mmol. Show the dynamics in the concentrations of insulin $C(t)$ and glucose $G(t)$, for two values of the feedback strength:

i. $K = 0.01$ L/min; and

ii. $K = 0.1$ L/min.

Explain what you observe.