Lecture 10: Fourier Analysis and Frequency Response

Tuesday, November 10, 2020 8:58 AM

References:

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 8 & Ch. 11 (Sec. 8.5; Sec. 11.1 - 11.4).

Laplace vs Fourier revisited (see Lecture 3):

$$\mathcal{L}(x(t)) = \int_{0}^{+\infty} x(t) \cdot e^{-st}\, dt = X(s)$$

$$F(x(t)) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$= \mathcal{L}(x(t)) \text{ where: } \begin{cases} s = j\omega \quad (\sigma = 0) \\ x(t) = 0 \text{ for all } t \leq 0 \\ \qquad I.C. \equiv 0 \end{cases}$$

$$\text{If } I.C. = 0: \quad X(s) = H(s) \cdot f(s)$$

$$X(j\omega) = H(j\omega) \cdot f(j\omega)$$

Transfer function in
Laplace and Fourier domains
($s = j\omega$)

Main differences between Laplace and Fourier (see Practice Quiz 1):

1) $\mathcal{L}: \quad 0 \leq t \leq +\infty \qquad + \text{I.C. @ } 0$

$\mathcal{F}: \quad -\infty \leq t \leq +\infty \qquad \text{No I.C.}$

2) $H(s) = H(j\omega) \quad \text{where } s = j\omega$

$\mathcal{L}: \quad s = \sigma + j\omega$

COMPLEX
--> exponentials

$\mathcal{F}: \quad s = j\omega$

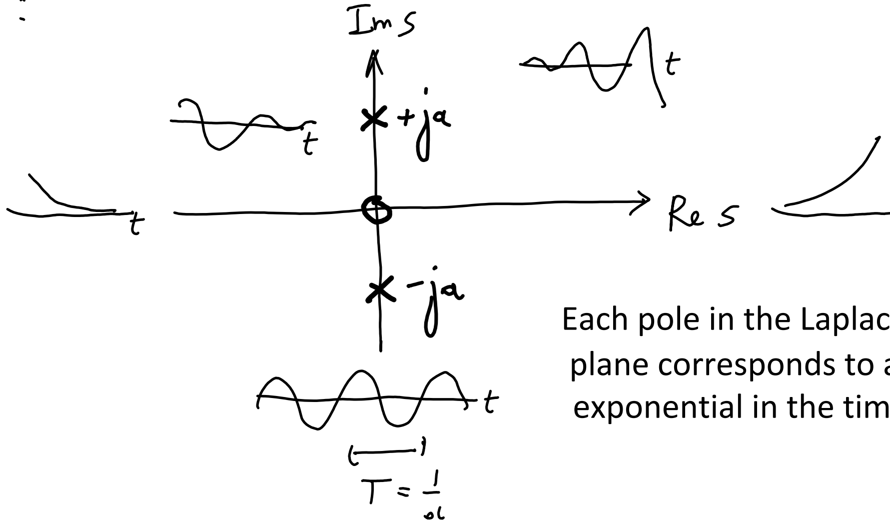
PURELY IMAGINARY
--> sines / cosines

$\mathcal{L}: \quad x(t) = e^{\sigma t} e^{j\omega t} + \text{c.c.}$

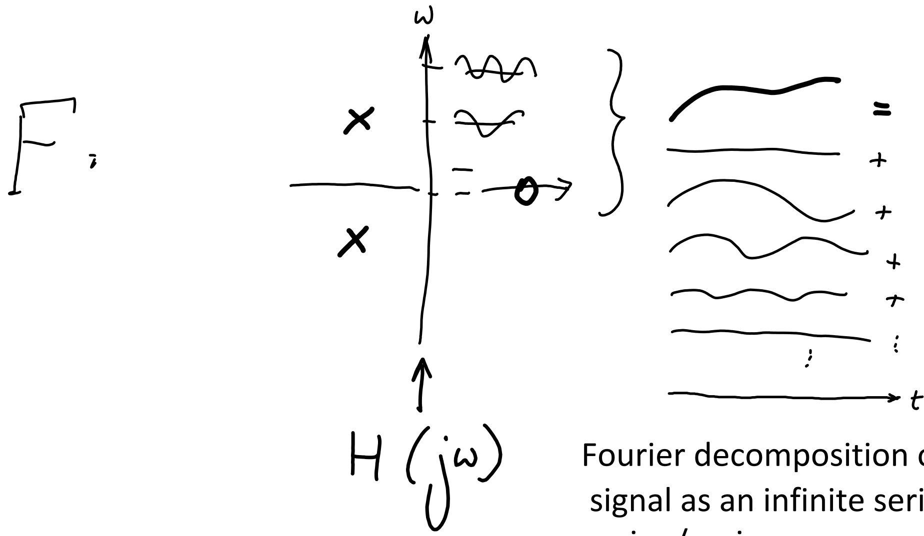
$\mathcal{F}: \quad x(t) = \underbrace{e^{j\omega t}}_{\cos(\omega t) + j\sin(\omega t)} + \text{c.c.}$

complex
conjugate

$\mathcal{L}:$



Each pole in the Laplace complex plane corresponds to a complex exponential in the time domain



Fourier decomposition of the signal as an infinite series of sine/cosine components

Frequency response of a linear time-invariant (LTI) system:

$$f(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow u(t)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(j\omega)\, e^{j\omega t}\, d\omega \qquad u(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(j\omega)\, e^{j\omega t}\, d\omega$$

Fourier decomposition of the input $f(t)$ as an infinite series of complex exponentials at radial frequency ω, each with complex amplitude $f(j\omega)$.

Corresponding Fourier decomposition of the output $u(t)$, where each output component $u(j\omega)$ is linearly related to the input component $f(j\omega)$ at the *same* radial frequency ω:

$$u(j\omega) = H(j\omega) \cdot f(j\omega)$$

The complex gain of this linear relationship is given by the transfer function of the system response at $s = j\omega$.

The *magnitude* of this complex gain scales the *amplitude* of the sinusoid from input to output, and the *angle* offsets the *phase* of the sinusoid from input to output:

$$\underbrace{H(j\omega)}_{\text{Transfer function (complex)}} = \underbrace{A(\omega)}_{\text{Amplitude (Gain)}} \cdot e^{j\overbrace{\varphi(\omega)}^{\text{Phase}}}$$

Single-frequency complex sinusoid:

$$e^{j\omega t} \longrightarrow A(\omega)\cdot e^{j(\omega t+\varphi(\omega))}$$

$$\cos\omega t + j\sin\omega t \longrightarrow A(\omega)\cdot\big(\cos(\omega t+\varphi(\omega)) + j\sin(\omega t+\varphi(\omega))\big)$$

Each frequency component:

$$f(j\omega) \longrightarrow A(\omega)\cdot e^{j\varphi(\omega)}\cdot f(j\omega)$$

$$A_f\, e^{j\varphi_f} \longrightarrow \underbrace{A(\omega)\cdot A_f}_{A_u}\; e^{j\overbrace{(\varphi(\omega)+\varphi_f)}^{}}$$

$$\varphi_u$$

Phases add

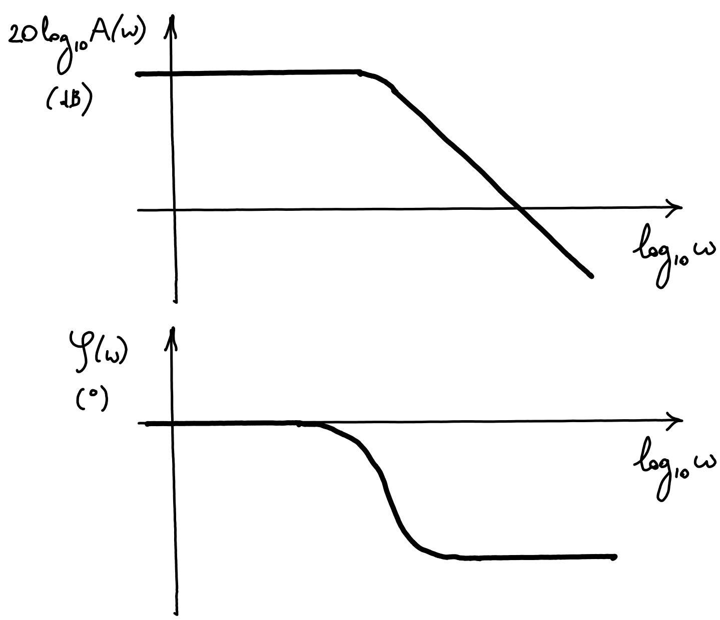
Amplitudes multiply (but add in their logarithms)

Bode frequency analysis:

GAIN (dB): $$20\log_{10} A_u = 20\log_{10} A(\omega) + 20\log_{10} A_f$$

PHASE (°): $$\underbrace{\varphi_u}_{\text{Output}} = \underbrace{\varphi(\omega)}_{\text{System response}} + \underbrace{\varphi_f}_{\text{Input}}$$

--> Bode plot
log amplitude & *phase* as a function of *log frequency, e.g.*:



Units:

Gain:

$$A(\omega) = |H(j\omega)|$$

[dB] : 20dB/dec
decibels 6dB/oct

Phase:

$$\varphi(\omega) = \angle H(j\omega)$$

[°] degrees
[rad] radians

Radial/cyclic frequency:

$$\omega = 2\pi f$$

$\left[\frac{rad}{s}\right]$ [Hz]