Lecture 11: Bode Design and Phase Margin

Thursday, November 12, 2020 8:48 AM

References:

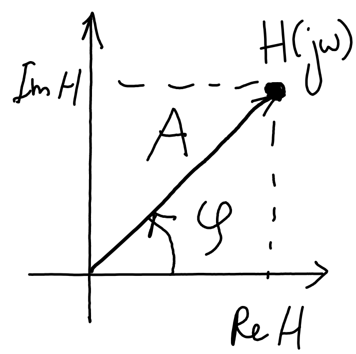
Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 8 & Ch. 9 (Sec. 8.5, 9.6, 9.7).

$$H(j\omega) = A(\omega) \cdot e^{j\varphi(\omega)}$$

Transfer function (complex)

Amplitude (Gain)

Phase



Bode plot: log amplitude & phase vs log (radial) frequency:

GAIN (dB):

$$20 \log_{10} A(\omega) = 20 \log_{10} |H(j\omega)|$$

$$= 10 \log_{10} |H(j\omega)|^2$$

$$= 10 \log_{10} \left(Re(H(j\omega))^2 + Im(H(j\omega))^2 \right)$$

PHASE (°):

$$\varphi(\omega) = \angle H(j\omega)$$

$$= \text{Arctan}\left(\frac{Im(H(j\omega))}{Re(H(j\omega))} \right)$$

Bode plot parameters, directly from the poles and zeros of the transfer function:

$$\underset{(s=j\omega)}{H(s)} = c\,\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

$$\log H(s) = \log c + \sum_{i=1}^{m} \log(s-z_i) - \underbrace{\sum_{j=1}^{n} \log(s-p_j)}_{+\sum_{j=1}^{n} \log \frac{1}{s-p_j}}$$

$$= \log A(s) + j\,\varphi(s)$$

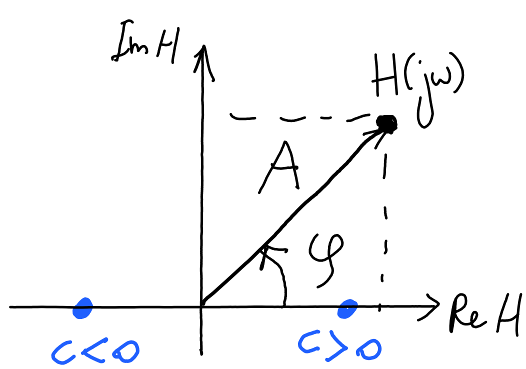
$$20\log_{10} A(\omega) = 20\log_{10}|c| + \sum_{i=1}^{m} 20\log_{10}|j\omega - z_i| - \sum_{j=1}^{n} 20\log_{10}|j\omega - p_j|$$

$$\varphi(\omega) = \angle c + \sum_{i=1}^{m} \angle\, j\omega - z_i - \sum_{j=1}^{n} \angle\, j\omega - p_j$$

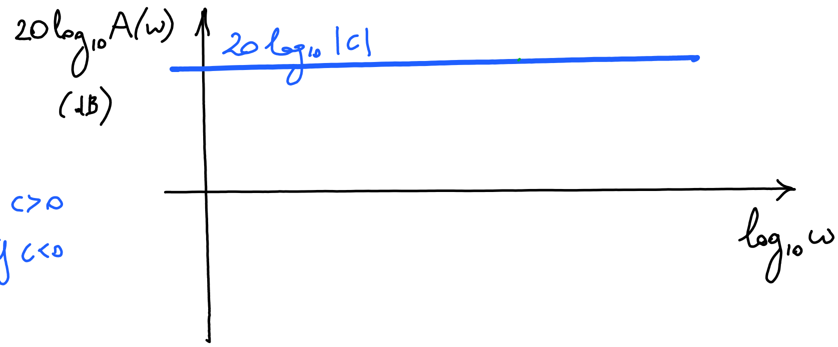
The Bode plot can be constructed by summing individual contributions for the constant, for each zero, and for each pole.

1. Constant *c*:

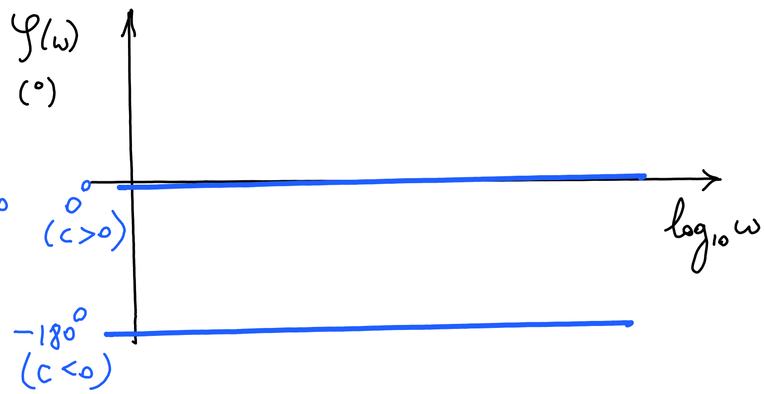
Real! Thus, either positive or negative:



$C:$



$$|c| = \begin{cases} c & \text{if } c>0 \\ -c & \text{if } c<0 \end{cases}$$



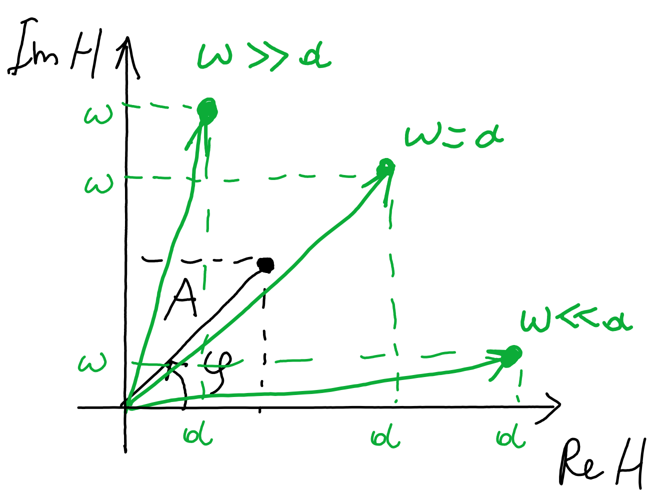
$$\angle c = \begin{cases} 0^\circ & \text{if } c>0 \\ (-)180^\circ & \text{if } c<0 \end{cases}$$

The constant contributes a flat response both in log amplitude and in phase.

2. Zero $(s + a)$:

ZERO @ $s = -\alpha$ $\quad \alpha > 0$ (<) $\quad \left(\alpha = \frac{1}{2}\right)$

$\alpha = 0$



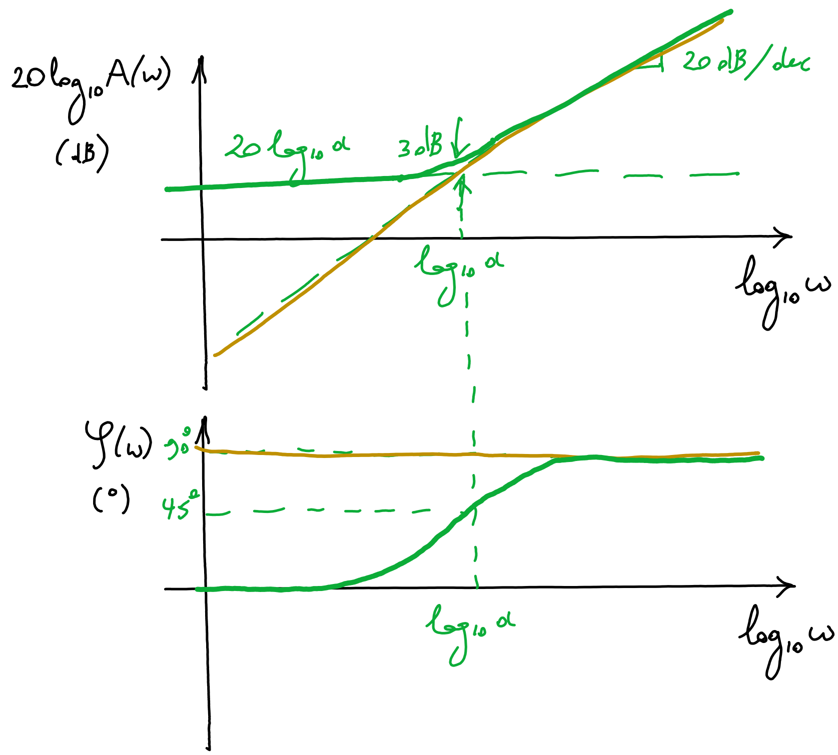
$$j\omega + \alpha : \quad 20\log_{10}|j\omega+\alpha| = 10\log_{10}(\omega^2+\alpha^2)$$

$$\approx 20\log_{10}\alpha \qquad \text{for } \omega \ll \alpha$$

$$\approx 20\log_{10}\alpha + \underbrace{10\log_{10}2}_{=3\,\mathrm{dB}} \qquad \text{for } \omega = \alpha$$

$$\approx 20\log_{10}\omega \qquad \text{for } \omega \gg \alpha$$

$$\angle j\omega+\alpha = \mathrm{Arctan}\left(\frac{\omega}{\alpha}\right) \approx \begin{cases} 0^\circ & \text{for } \omega \ll \alpha \\ 45^\circ & \text{for } \omega = \alpha \\ 90^\circ & \text{for } \omega \gg \alpha \end{cases}$$



To first order, each **zero** (@ s = -a) contributes a **20 dB/dec rise in slope** of log amplitude, and a **+90º step** in phase, for radial frequencies **above** the magnitude of that zero ($\omega > |a|$).

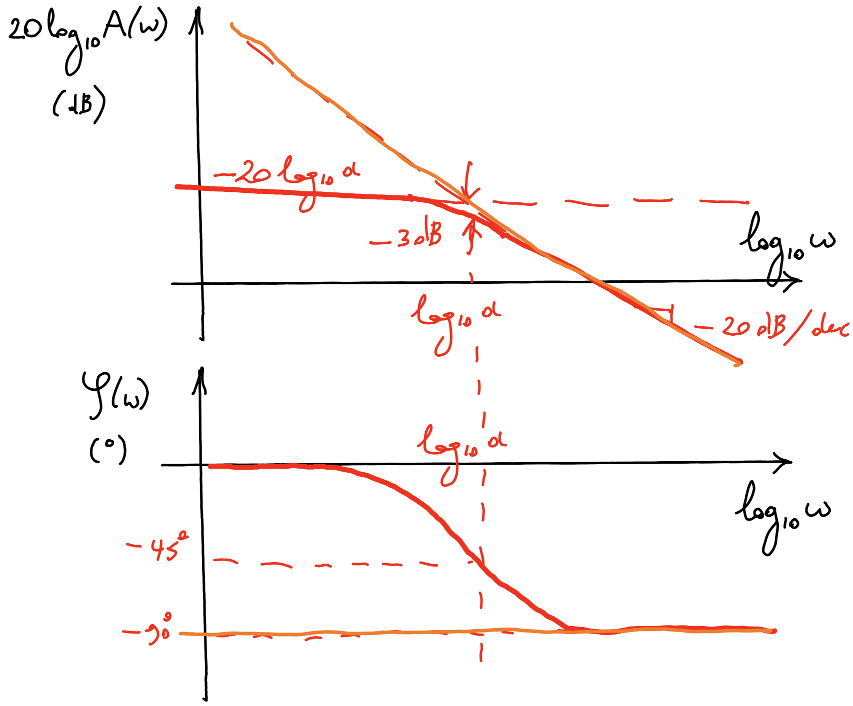
More precisely, for a real negative zero, at the transition ($\omega = a$), the log amplitude has risen by +3 dB, and the phase has stepped halfway at 45º.

3. Pole $1 / (s + a)$:

POLE @ $s = -\alpha$ $\alpha > 0$ $(\alpha = \frac{1}{2})$

0 $\alpha = 0$

Same as ZERO @ $s = -a$, except for sign reversal both in log amplitude and phase:



To first order, each **pole** (@ $s = -a$) contributes a **-20 dB/dec decline in slope** of log amplitude, and a **-90º step** down in phase, for radial frequencies **above** the magnitude of that zero ($\omega > |a|$).

More precisely, for a real negative pole, at the transition ($\omega = a$), the log amplitude has fallen by -3 dB, and the phase has stepped halfway at -45º.

Putting it all together: constructing the Bode plot from individual contributions:

1. **Start from log amplitude and phase** (0 or 180^o) **given by the constant *c*.**

2. **Rank order the magnitudes *a* of all zeros and poles, from lowest** (possibly zero) **to highest. For each *a*:**

   a. **If it is a zero: increase slope of log amplitude by +20 dB/dec, and step phase by +90^o, for all frequencies above *a***, with a +3 dB rise and 45^o midway step at the transition frequency *a*.

   b. **If it is a pole: decrease slope of log amplitude by -20 dB/dec, and step phase down by -90^o, for all frequencies above *a***, with a -3 dB fall and -45^o midway step at the transition frequency *a*.

3. **If needed, fix indeterminate offset in log amplitude by taking limits of the transfer function for zero, infinite, or mid-band frequency where the log amplitude slope is flat.**

Notes:

1. The slope of log amplitude, in units 20 dB/dec, is almost always, but only approximately, equal to the phase, in units 90^o. It is always good to check consistency between the two.

2. The above assumes all zeros and poles are real and negative. For pairs of complex conjugate zeros (poles) with magnitude *a*, both contribute for a total (-)40dB/dec slope in log amplitude and (-)180^o step in phase, with (-)90^o midway step at resonance frequency *a*. Log amplitude at the resonance frequency depends on damping, and approaches zero (infinity) for zero damping. Plug in values in the transfer function at resonance to find log amplitude at resonance.

Example:

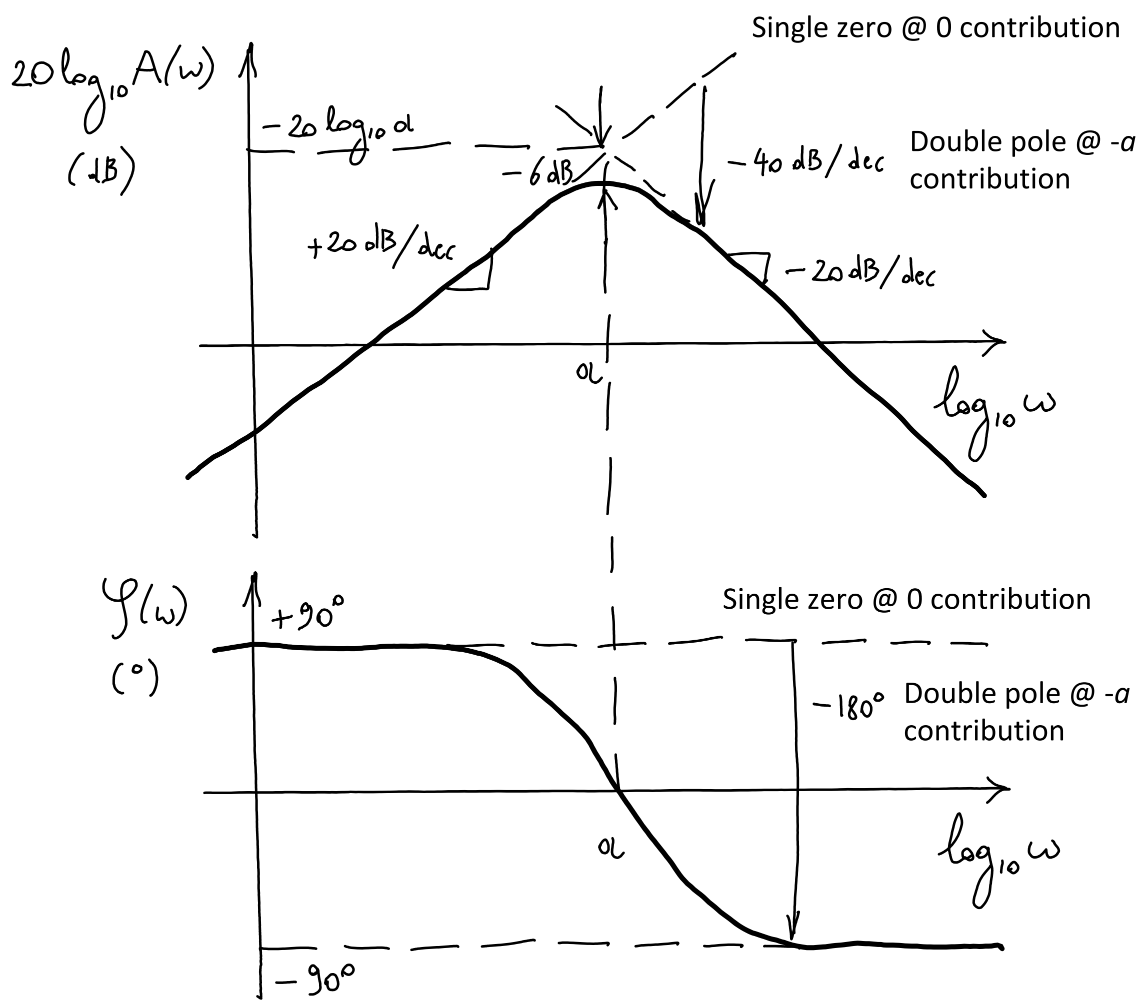
$$H(s) = \frac{s}{(s+a)^2}$$

Single zero @ $s = 0$

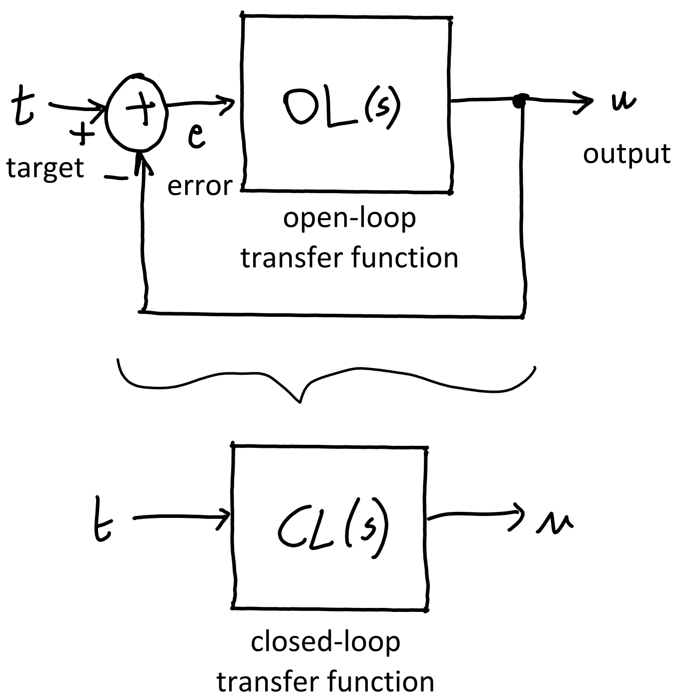
Two poles @ $s = -a$

$$H(j\omega) = \frac{j\omega}{(j\omega + a)^2}$$

$$@\ \omega = a:\quad \frac{ja}{\left((j+1)a\right)^2} = \frac{\cancel{j}\,\cancel{a}}{2\cancel{j}\,a^{\cancel{2}}} = \frac{1}{2a}$$



Closing the loop: open-loop phase margin for closed-loop stability



Open-loop Bode design:

controller   biosystem   (including measurement)

$$OL(s) = F(s) \cdot H(s) \cdot G(s)$$

$$OL(j\omega) = F(j\omega) \cdot H(j\omega) \cdot G(j\omega)$$

$$A_{OL}(\omega) = A_F(\omega) \cdot A_H(\omega) \cdot A_G(\omega)$$

$$20 \log A_{OL} = 20 \log A_F + 20 \log A_H + 20 \log A_G$$

$$\varphi_{OL}(\omega) = \varphi_F(\omega) + \varphi_H(\omega) + \varphi_G(\omega)$$

Amplitudes multiply.
Log amplitudes and phases add.

Closed-loop stability:

$$u(s) = OL(s) \cdot \left(t(s) - u(s)\right)$$

$$CL(s) = \frac{u(s)}{t(s)} = \frac{OL(s)}{1 + OL(s)} \longrightarrow 1 \quad \text{as} \quad OL \to \infty$$

$$\approx \begin{cases} 1 & \text{for} \quad OL(s) \gg 1 \\ OL(s) & \text{for} \quad OL(s) \ll 1 \end{cases}$$

Poles: $OL(s) = -1 \longrightarrow \begin{array}{l} A_{OL}(\omega) = 1 \\ \varphi_{OL}(\omega) = -180^\circ \end{array}$

The **closed-loop system** is **unstable** and oscillates at any frequency ω where:

1) the **open-loop gain is 1 (0 dB)** in magnitude, and
2) the **open-loop phase is -180°** (anti-phase).

Phase margin is the degree to which the open-loop phase is higher than -180° at the frequency ω where the open-loop gain is 1 (0 dB) in magnitude.

Conversely, **gain margin** is the amount (in dB) of attenuation in open-loop gain at the frequency where the open-loop phase is -180°.

For **zero** or *negative* phase (gain) margin, the closed-loop system is unstable exhibiting **unbounded** or *rising* oscillations. The higher the phase (gain) margin, the more damped the closed-loop dynamics. For phase margins of -90° and above, the closed-loop dynamics is overdamped, with approximately a first-order response with time constant $\tau = 1 / \omega$.

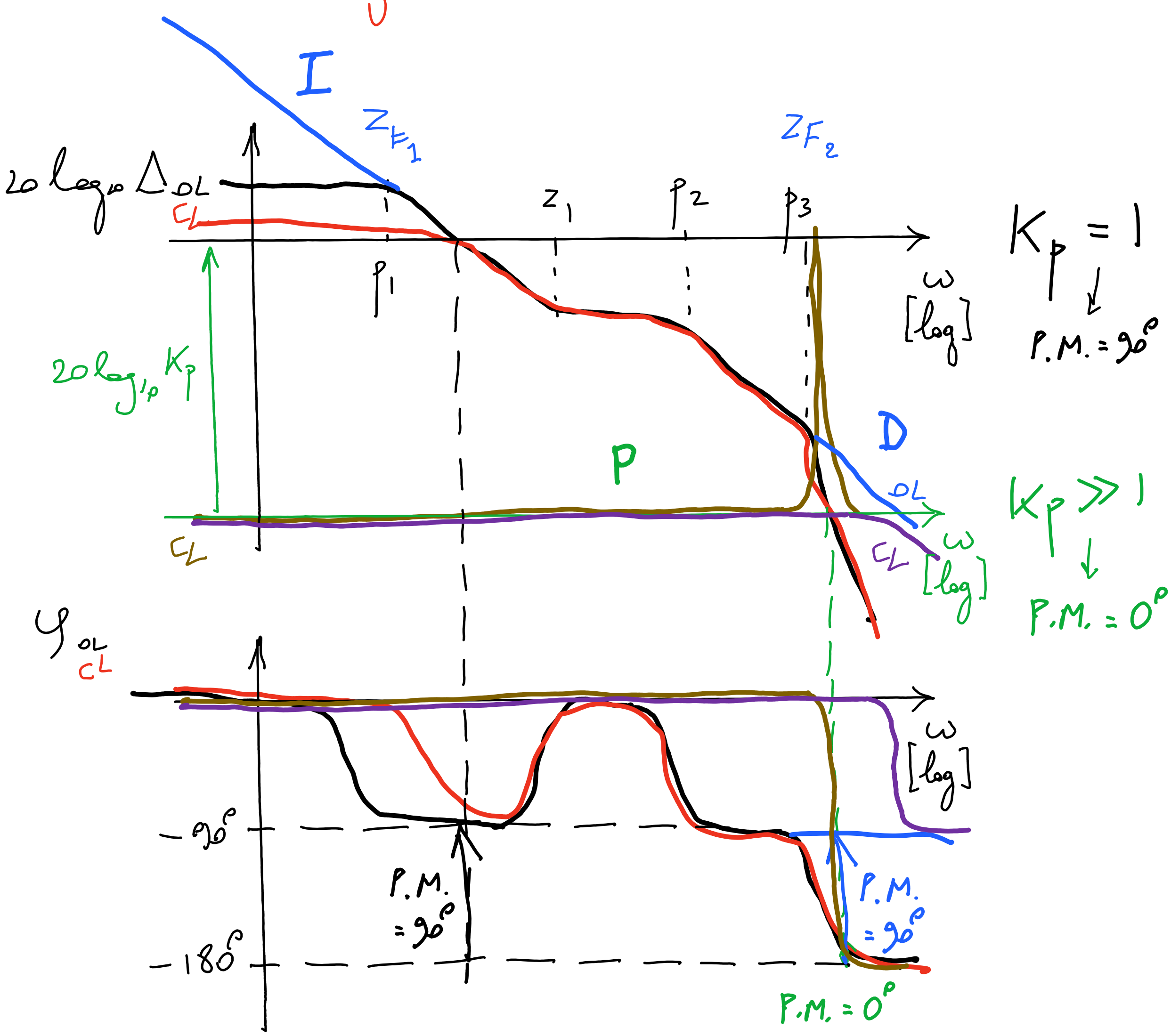
Bode stability analysis and design, e.g.:

$$H(s) = \frac{s + z_1}{(s+p_1)(s+p_2)(s+p_3)} \quad ; \quad p_1 < z_1 < p_2 < p_3$$

zero @ s = $-z_1$
poles @ s = $-p_1$, $-p_2$, $-p_3$

$$OL(j\omega) = A_{OL}(\omega) \cdot e^{j\varphi_{OL}(\omega)}$$

$$CL(j\omega) = A_{CL}(\omega) \cdot e^{j\varphi_{CL}(\omega)}$$



Bode design of proportional-integral-derivative (PID) control:

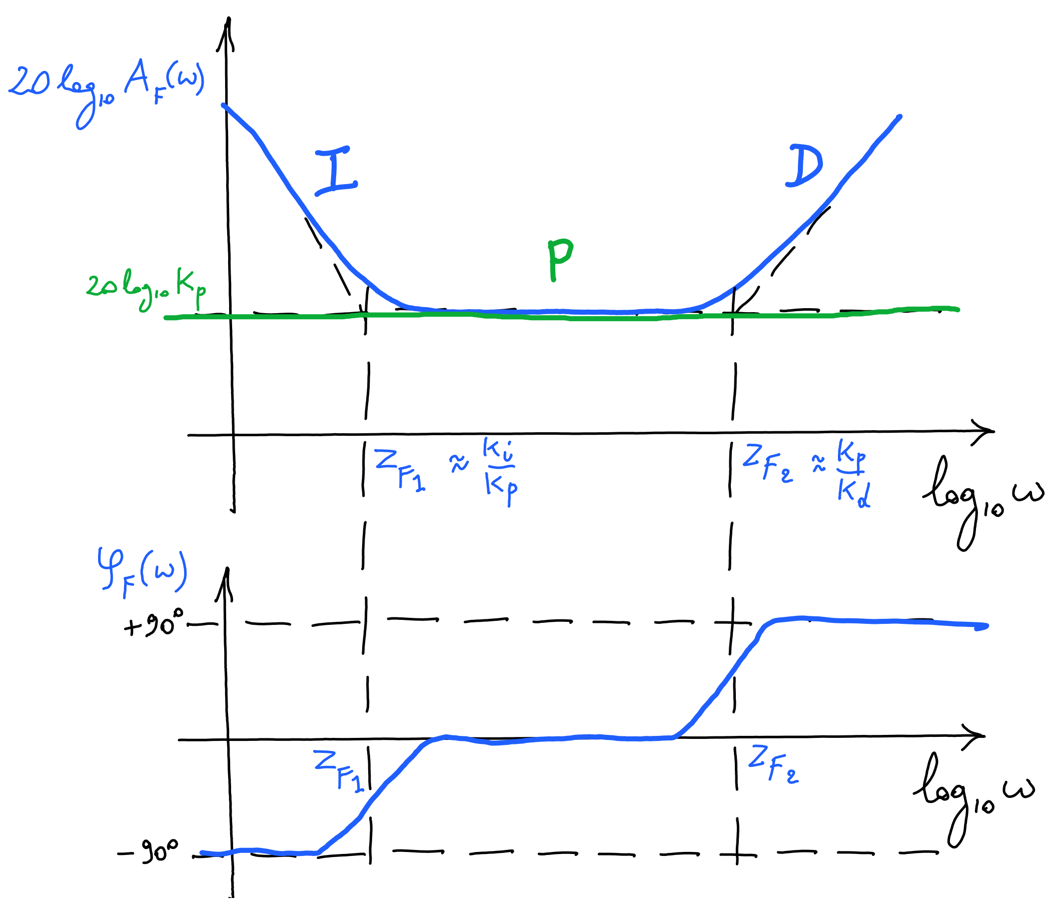
$$F(s) = K_p + K_i \frac{1}{s} + K_d s$$

$$= \frac{K_d s^2 + K_p s + K_i}{s}$$

--> 2 zeros @ $s = -z_{F1}, -z_{F2}$

--> 1 pole @ $s = 0$

$$\approx \begin{cases} \dfrac{K_i}{j\omega} & \text{for} \quad \omega < z_{F_1} \quad \text{low frequency} \\ K_p & \text{for} \quad z_{F_1} < \omega < z_{F_2} \quad \text{midband frequency} \\ K_d\, j\omega & \text{for} \quad \omega > z_{F_2} \quad \text{high frequency} \end{cases}$$



Proportional control increases open-loop gain to bring closed-loop gain closer to unity for **low error at low frequencies**, but **decreases open-loop phase margin** potentially causing instability with **oscillation at high frequency**.

Integral control further **boosts open-loop gain by 20 dB/dec at low frequencies** to **decrease low-frequency closed-loop error**.

Derivative control increases open-loop phase margin by 90^o at high frequencies to **improve high-frequency closed-loop dynamics**.