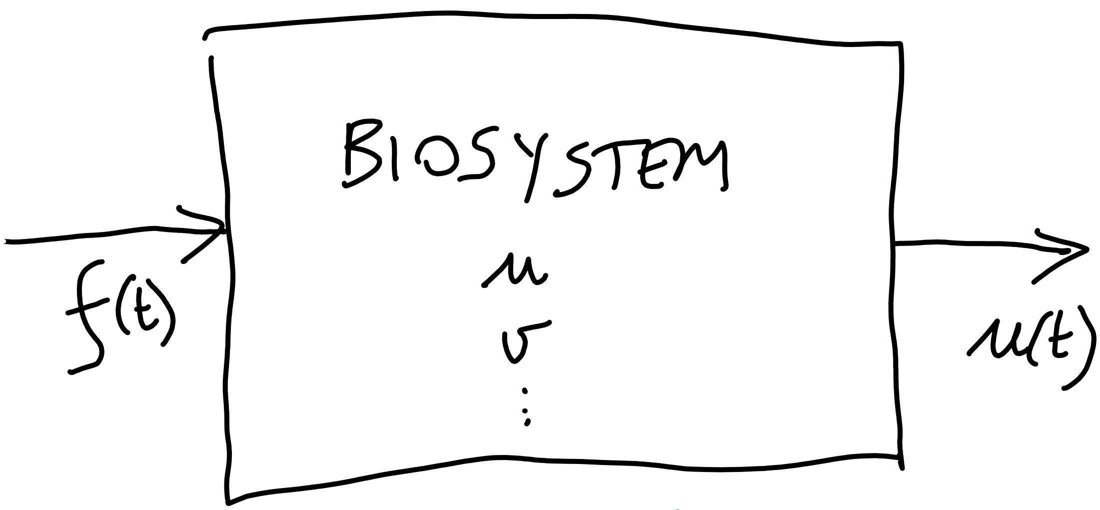
Lecture 2: Linear models

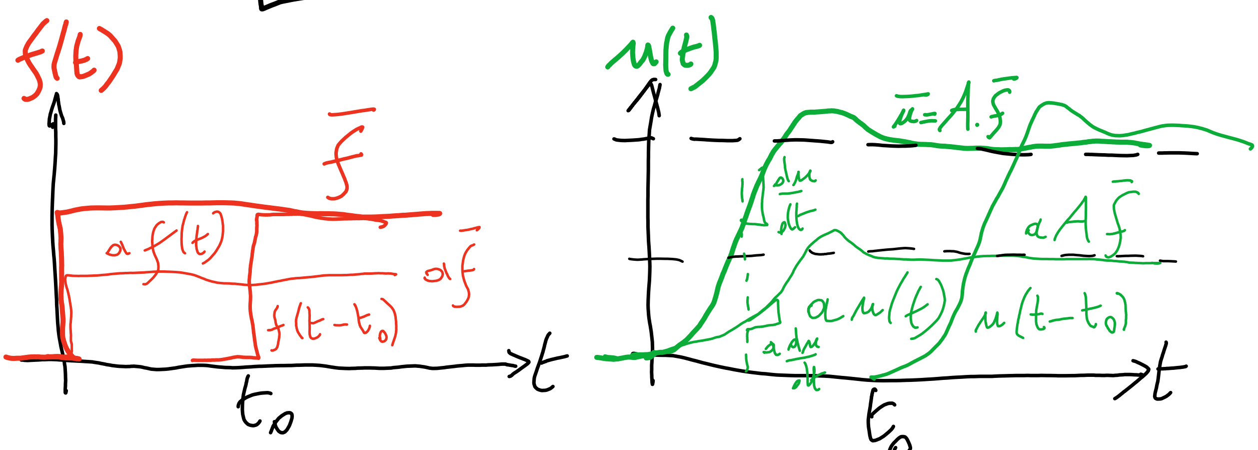
Tuesday, October 6, 2020 8:48 AM

References:

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 2 (Sec. 2.12, 2.13) and Ch. 3 (Sec. 3.1 - 3.4).

Linear time-invariant systems:





a: linear scaling
t_0: time shift invariance

A: biosystem response gain

$$\begin{cases} \dfrac{du}{dt} = F\left(u(t), v(t), f(t), t\right) \\ \dfrac{dv}{dt} = G\left(u(t), v(t), f(t), t\right) \end{cases}$$

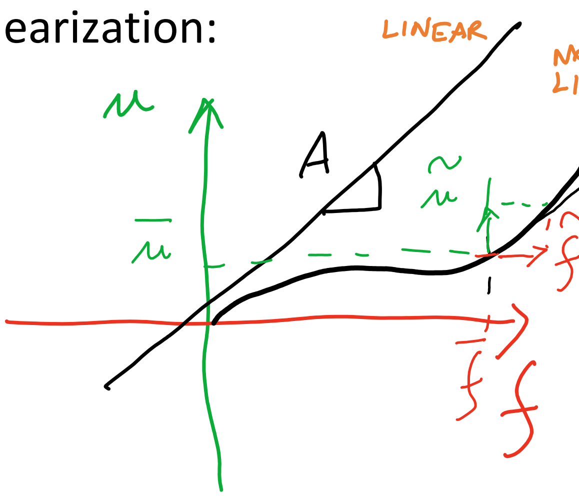
LINEAR: $\begin{cases} \dfrac{du}{dt} = a\,u + b\,v + c f \\ \dfrac{dv}{dt} = d\,u + e\,v + g \cdot f \end{cases}$

LTI: $a, b, c \ldots g$ = constant

In matrix-vector form:

$$\begin{cases} \dfrac{d\vec{x}}{dt} = \bar{\bar{A}} \cdot \vec{x} + \vec{b} \cdot f \\ u = \vec{c} \cdot \vec{x} \end{cases} \quad \text{with } \vec{x} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Linearization:



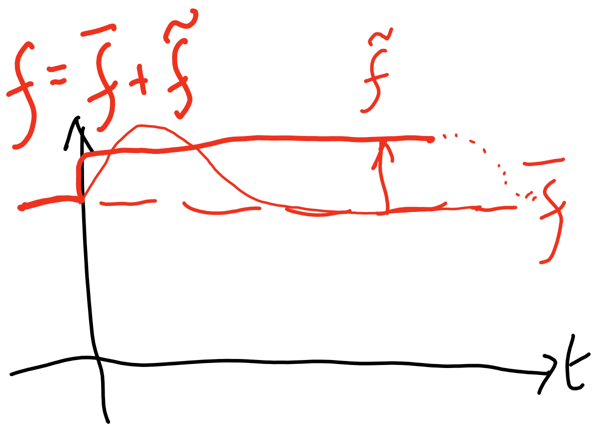
$$\tilde{u} = \overbrace{\left.\frac{du}{df}\right|_{\bar{f}}}^{A_{effective}} \cdot \tilde{f}$$

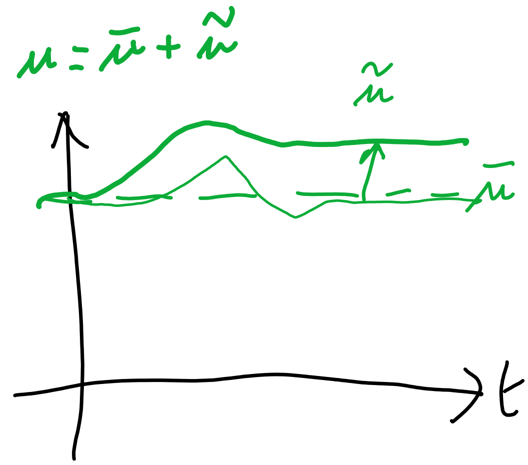
A nonlinear system is approximately linear for small-signal changes around the steady-state operating point.

Superposition: decompose variables f, u, v, … as a sum of two parts:

1) steady-state background defining the operating point: $\bar{f}, \bar{u}, \bar{v}, \ldots$ NONLINEAR

2) small-signal transient fluctuations around steady-state: $\tilde{f}, \tilde{u}, \tilde{v}, \ldots$ LINEARIZED





$$F(u, v, f, t) \approx \underbrace{\left.\frac{\partial F}{\partial u}\right|_{\underbrace{\bar{f}, \ldots}_{ss}}}_{a} \cdot \tilde{u} + \underbrace{\left.\frac{\partial F}{\partial v}\right|_{ss}}_{b} \tilde{v} + \underbrace{\left.\frac{\partial F}{\partial f}\right|}_{c} \tilde{f}$$

1st order

Taylor expansion for small-signal excursions around the steady-state operating point

0th order $\quad + \cancel{F(\bar{u}, \bar{v}, \bar{f}, t)}$

$= 0$ @ S.S.

steady-state

$\cancel{+ \text{H.O.T}}$

$$\begin{cases} \dfrac{du}{dt} = F\big(u(t), v(t), f(t), t\big) = \cancel{\dfrac{d\bar{u}}{dt}} + \dfrac{d\tilde{u}}{dt} \\[2ex] \dfrac{dv}{dt} = G\big(u(t), v(t), f(t), t\big) \quad \text{SIMILAR} \end{cases}$$

LINEAR:

LINEARIZED:

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = a\tilde{u} + b\tilde{v} + c\tilde{f} \quad \cancel{+\,\text{H.o.T.}} & a = \left.\dfrac{\partial F}{\partial u}\right|_{s.s.} \text{ etc.} \\[2ex] \dfrac{d\tilde{v}}{dt} = d\tilde{u} + e\tilde{v} + g\cdot\tilde{f} & d = \left.\dfrac{\partial G}{\partial u}\right|_{s.s.} \text{ etc.} \end{cases}$$

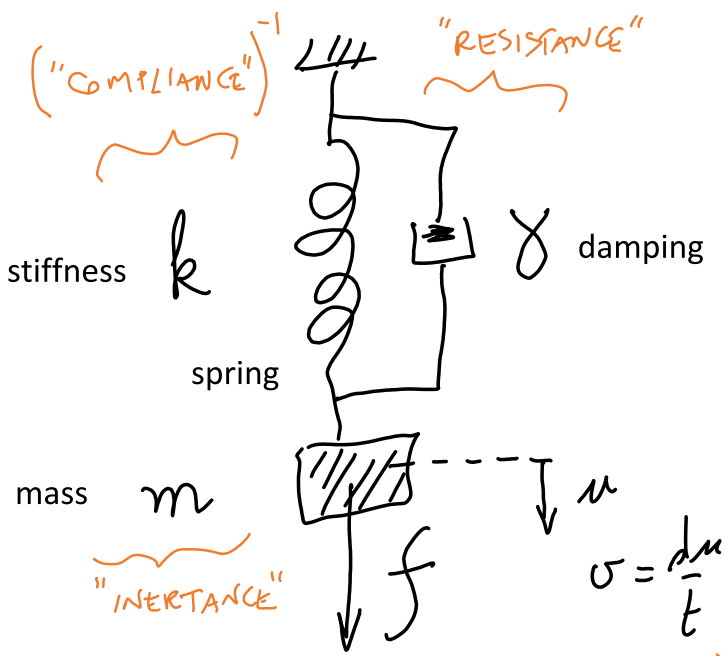
Conservative models:

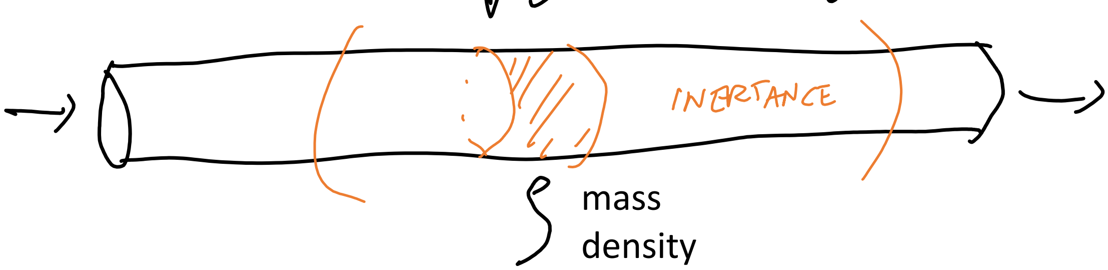
Conservation of momentum

$$m \cdot v = cst \quad \text{when} \quad f_{all} = 0$$

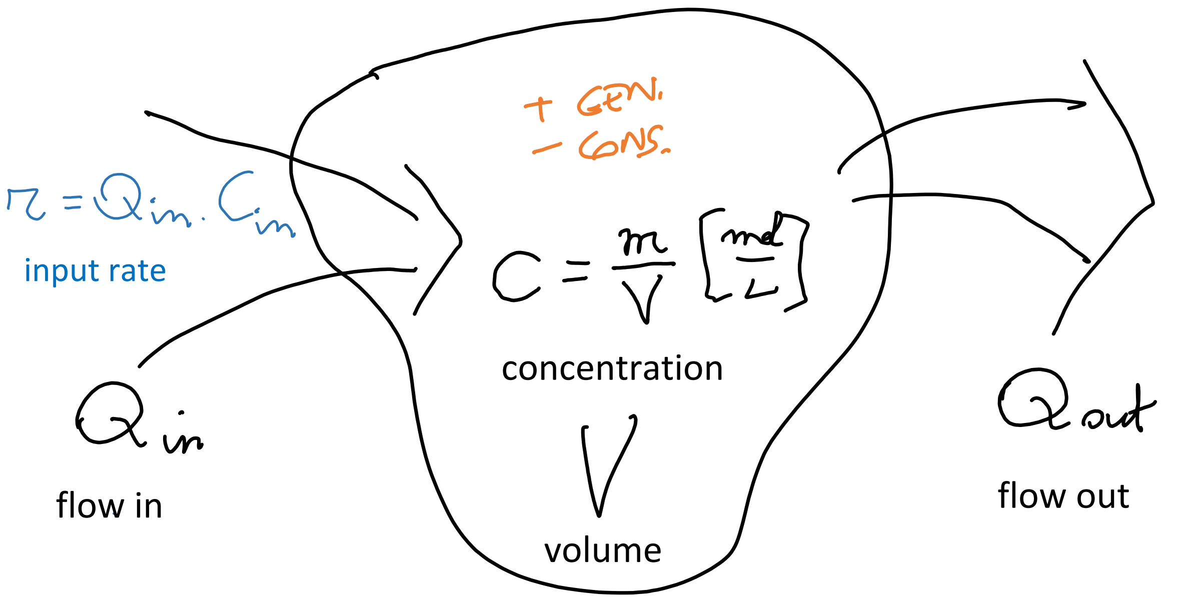
$$\frac{d}{dt}(mv) = f_{all} \qquad \text{Newton's law}$$

$$\begin{cases} \dfrac{du}{dt} = v \\[2ex] \dfrac{dv}{dt} = \dfrac{1}{m} \cdot f_{all} = \dfrac{1}{m} \cdot \left(f - ku - \gamma . v \right) \end{cases}$$





Conservation of mass ("stuff") -- flow in/out; generation



$$\frac{d}{dt} V = Q_{in} - Q_{out}$$

$$\left[\frac{1}{s}\right] [L] \qquad \left[\frac{L}{s}\right]$$

$$\frac{dC}{dt} = \underbrace{\frac{1}{V} r}_{\text{INPUT RATE}} \overbrace{- \frac{1}{V} Q_{out}}^{\frac{1}{\tau}} \cdot C \qquad \tau = \frac{V}{Q_{out}}$$

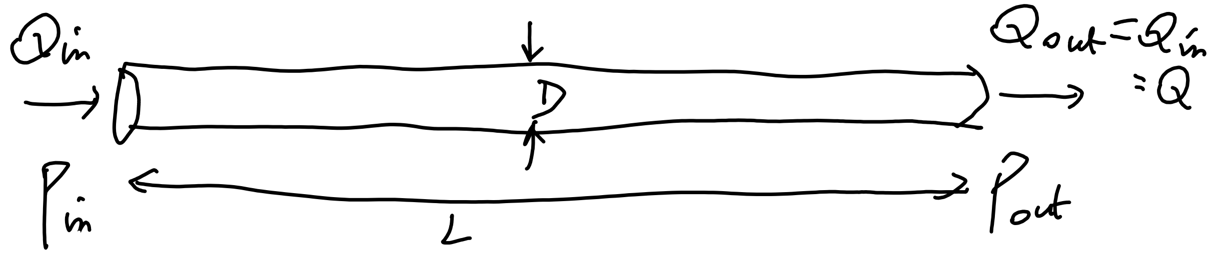
$$\frac{1}{s} \frac{mol}{L} \qquad \left[\frac{1}{L}\right]\left[\frac{mol}{s}\right] \qquad \left[\frac{1}{L}\right]\left[\frac{L}{s}\right]\left[\frac{mol}{L}\right]$$

+ GENERATION

− CONSUMPTION

Conservation of charge -- resistance, capacitance/compliance

Resistance of a tube (blood vessel, etc):



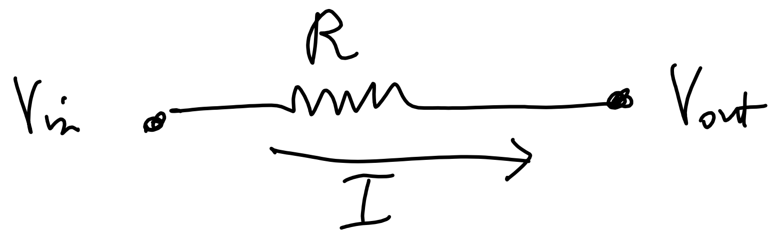
$$\Delta P = P_{out} - P_{in} = R \cdot Q$$

$R \uparrow$ if $L \uparrow$

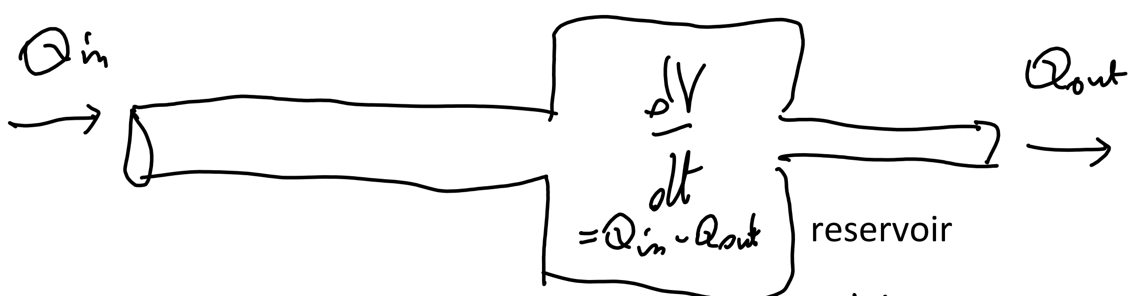
$R \downarrow$ if $D \uparrow$

electrical circuit equivalent (electrical potential and current):

$$\Delta V = V_{out} - V_{in} = R \cdot I$$



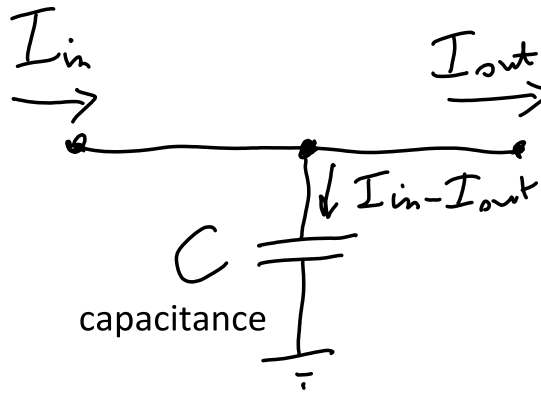
Compliance of a reservoir (organ, vessel wall elasticity, etc):



$$C \, \Delta P = \Delta V \quad \text{or} \quad C = \frac{\Delta V}{\Delta P}$$ compliance

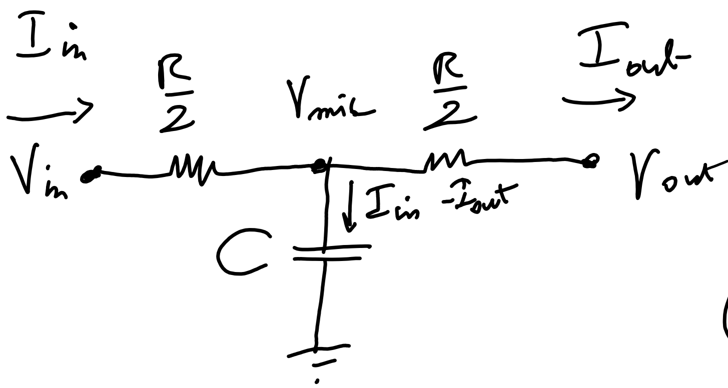
$$C \frac{dP}{dt} = \frac{dV}{dt} = Q_{in} - Q_{out}$$

electrical circuit equivalent:



$$C\frac{dV}{dt} = I_{in} - I_{out}$$

Resistance and capacitance/compliance combined (lumped electrical model):



$$\begin{cases} V_{in} - V_{mid} = \frac{R}{2} \cdot I_{in} \\ V_{mid} - V_{out} = \frac{R}{2} I_{out} \\ C\frac{dV_{mid}}{dt} = I_{in} - I_{out} \end{cases}$$

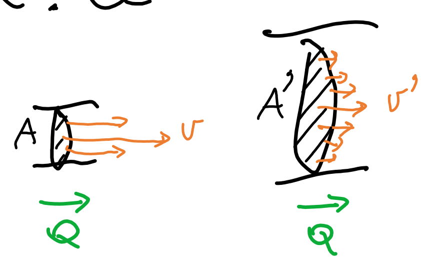
Conservation of energy

$$\Delta E = \Delta E_{pot} + \Delta E_{kin} = \Delta E_{heat}$$

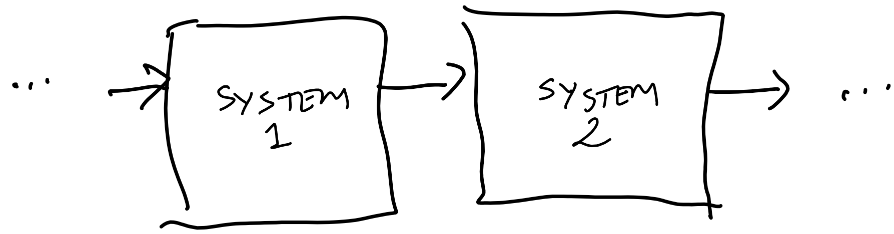
$$= V\Delta P + m \cdot g \cdot \Delta h + \frac{m}{2}\Delta(v^2)$$

$$= V\underbrace{\left(\Delta P + \rho g \Delta h + \frac{\rho}{2}\Delta(v^2)\right)}_{= R \cdot Q}$$

$$Q = \frac{\overbrace{\pi D^2}^{A}}{4} \cdot v$$



State and compartment models:



Modular cascade of subsystems, each representing one compartment in the model.

Chemical reagents:

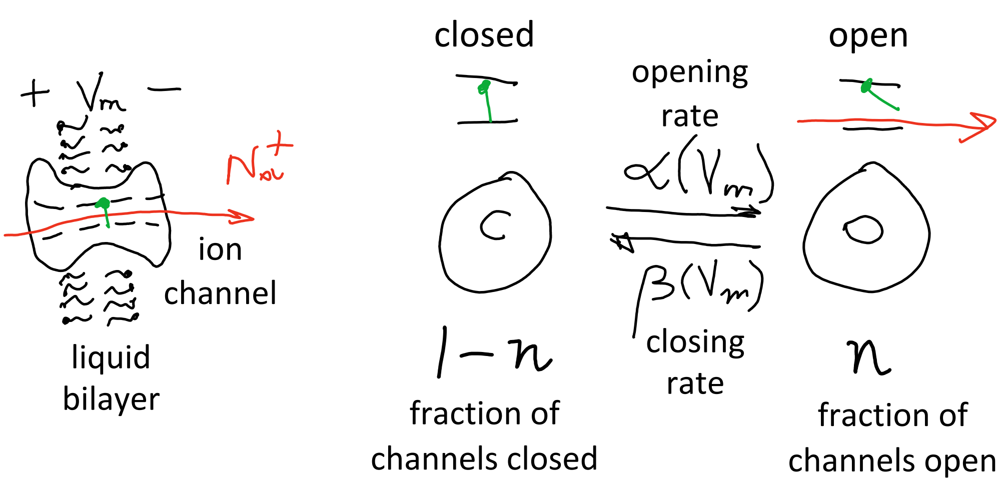
$$A + B \underset{k_r}{\overset{k_f}{\rightleftharpoons}} C$$

Rate kinetics:

$$\frac{d[C]}{dt} = -\frac{d[A]}{dt} = -\frac{d[B]}{dt} = k_f [A].[B] - k_r [C]$$

Nonlinearity due to the product turns almost linear when one of the concentrations dominates and remains relatively constant.

Ion channels:



Rate kinetics:

$$\frac{dn}{dt} = \alpha(V_m)(1-n) - \beta(V_m)\, n$$