Lecture 3: Linear transforms

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References:

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 4 (Sec. 4.1 - 4.8).

LTI ODE Systems: the power of exponentials

$$f(t) \longrightarrow \boxed{LTI} \longrightarrow x(t)$$

$$c f(t) \longrightarrow c x(t)$$

$$f_1(t) + f_2(t) \xrightarrow{\text{linear}} x_1(t) + x_2(t)$$

$$f(t-t_0) \xrightarrow{\text{time-invariant}} x(t-t_0)$$

$$\underbrace{\sum_i c_i f_i(t)}_{\text{LINEAR TRANSFORM}} \longrightarrow \sum_i c_i x_i(t)$$

Any linear transformation of solutions is still a solution to the LTI ODE. We will seek linear transforms that permit finding solutions using just linear algebra.

$$\frac{df}{dt} \xrightarrow{\quad\quad} \frac{dx}{dt}$$

LTI ODE

$$\frac{d^i f}{dt^i} \xrightarrow{\quad\quad} \frac{d^i x}{dt^i}$$

Derivatives are also linear operators, and are invariant to linear transforms.

Linear transforms acting on LTI ODEs:

LTI ODE in explicit form:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_0 x(t) = f(t)$$

LTI ODEs in canonical form (equivalent):

$$\frac{d}{dt} \vec{x} = \bar{\bar{A}} \cdot \vec{x}(t) + \vec{b} \cdot f(t)$$

$$\vec{x} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

state vector

e.g.:

$$x_1 = x$$
$$x_2 = \frac{d}{dt} x_1 = \frac{d}{dt} x$$
$$x_3 = \frac{d}{dt} x_2 = \frac{d^2}{dt^2} x$$
$$\vdots$$
$$x_n = \frac{d}{dt} x_{n-1} = \frac{d^{n-1}}{dt^{n-1}} x$$

or:

$$\frac{d}{dt} x_1 = x_2$$
$$\frac{d}{dt} x_2 = x_3$$
$$\vdots$$
$$\frac{d}{dt} x_{n-1} = x_n$$
$$\frac{d}{dt} x_n = -\frac{a_{n-1}}{a_n} x_n \dots - \frac{a_0}{a_n} x_1 - \frac{1}{a_n} f(t)$$

Initial conditions are required to specify the solution:

$$I.C.: \quad \vec{x}(0) = \vec{x}_0$$

$$\begin{cases} x(0) = x_0 \\ \frac{d}{dt} x(0) = x_0^{(1)} \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}} x(0) = x_0^{(n-1)} \end{cases}$$

"Calculus textbook" solution to the homogeneous LTI ODE (f = 0):

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_0 x(t) = 0$$

Try: $x(t) = e^{st}$

$$\frac{d^n}{dt^n} x(t) = s^n e^{st} = s^n x(t)$$

Exponentials are special! Derivatives reduce to scaling factors polynomial in *s*, turning the problem of solving the ODE into finding roots of a polynomial.

$$\left(a_n s^n + a_{n-1} s^{n-1} + \dots a_0\right) \underbrace{e^{st}}_{x(t)} = 0$$

Characteristic equation:

$$a_n s^n + a_{n-1} s^{n-1} + \dots a_0 = 0$$

n roots $s = s_i$ of this characteristic equation all give valid solutions.
The general homogeneous solution is any linear combination:

$$x(t) = \sum_i c_i e^{s_i t} \longrightarrow x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \underbrace{(\dots)}_{x(s)} e^{st} \, ds$$

LTI ODE homogeneous solution as finite sum of exponentials at discrete values $s = s_i$.
Specific values of the coefficients are determined by initial conditions.

Laplace extends this concept by expressing the signal $x(t)$ as an *infinite sum* (*integral*) of exponentials at continuous (complex) values s.
The coefficients in the expansion are values of the Laplace transform, $x(s)$.

Laplace transform

$$x(s) = \int_{0}^{+\infty} x(t)\, e^{-st}\, dt$$

Laplace expresses the time-varying signal $x(t)$ as a linear combination of exponentials e^{st}, with coefficients given by $x(s)$.

You can think of the complex Laplace variable $s = \sigma + j\,\omega$ as the coefficient of exponential time dependence in the signal, where σ is the rate of rise/decay, and ω is the angular frequency of oscillation in the signal over time.

The Laplace transform is *linear*, and *time-invariant*. It preserves linear scaling of the signal, and it turns a uniform delay t_0 into a common scaling factor e^{-st_0}.

It turns LTI ODEs into algebraic equations that are readily solved using just linear algebra, with coefficients that are polynomial in s.

Properties of Laplace transforms:

$$L(x(t)) = \int_{0}^{+\infty} x(t)\, e^{-st}\, dt$$

$$L(c_1 x_1(t) + c_2 x_2(t)) = c_1 L(x_1(t)) + c_2 L(x_2(t))$$

$$\frac{d}{dt}(uv) = \frac{du}{dt} v + u \frac{dv}{dt}$$

$$\int \frac{du}{dt} v\, dt = uv - \int u \frac{dv}{dt}\, dt$$

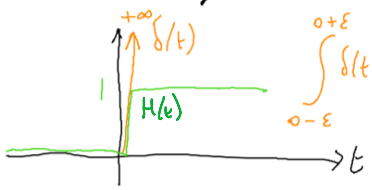
$$L\left(\frac{d}{dt} x(t)\right) = \int_{0}^{+\infty} \underbrace{\frac{d}{dt} x(t)}_{\frac{d}{dt} u} \cdot \underbrace{e^{-st}}_{v}\, dt$$

$$= \Big[x(t)\, e^{-st} \Big]_{0}^{+\infty} - \int_{0}^{+\infty} x(t) \cdot \left(-s \cdot e^{-st}\right) dt$$

$$0 - x(0) + s \underbrace{\int_{0}^{+\infty} x(t)\, e^{-st}\, dt}_{L(x(t))}$$

Laplace transforms of common functions:

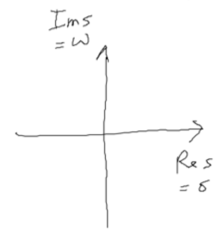
$$\mathcal{L}(\delta(t)) = \int_0^{+\infty} \delta(t)\, e^{-st}\, dt \approx \int_{0-\varepsilon}^{0+\varepsilon} \delta(t)\cdot 1\, dt = 1$$



$$\int_{0-\varepsilon}^{0+\varepsilon} \delta(t)\, dt = 1$$

$$e^{st} = e^{\sigma t}\cdot e^{j\omega t}$$

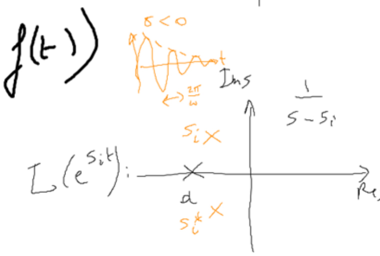
$$s = \sigma + j\omega$$



$$\mathcal{L}(H(t)) = \frac{1}{s}$$

$$\mathcal{L}(f(t-t_0)) = e^{-st_0}\cdot \mathcal{L}(f(t))$$

$$\mathcal{L}(e^{\alpha t}) = \frac{1}{s-\alpha}$$



Solving LTI ODEs using Laplace:

$$\mathcal{L}\left(\frac{d^i}{dt^i}x(t)\right) = s^i\cdot \mathcal{L}(x(t)) - \frac{d^{i-1}}{dt^{i-1}}x(0) - s\frac{d^{i-2}}{dt^{i-2}}x(0) - \dots s^{i-1}x(0)$$

$$\mathcal{L}\left(a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots a_0 x(t) = f(t)\right)$$

some polynomial in *s*

$$(a_n s^n + a_{n-1}s^{n-1} + \dots a_0)\underbrace{\mathcal{L}(x(t))}_{X(s)} - \underbrace{(\dots)\,p(s)}_{I.C.} = \underbrace{\mathcal{L}(f(t))}_{F(s)}$$

$$X(s) = \frac{F(s) + (\dots)\,p(s)}{a_n s^n + \dots a_0} = \frac{c_1}{s-s_1} + \frac{c_2}{s-s_2} + \dots$$

$$x(t) = c_1 e^{-s_1 t} + c_2 e^{-s_2 t} \dots$$

Inverse Laplace transform, using partial fractional decomposition:

$$x(s) = e\frac{1}{s} + g + \sum_i c_i \frac{1}{s-s_i}$$

$$\downarrow$$

$$x(t) = \underbrace{e\,H(t) + g\cdot\delta(t)}_{\text{DRIVE + I.C.}} + \sum_i c_i e^{+s_i t}$$

Fourier transform

$$F(x(t)) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j\omega t}\,dt$$

$$= L(x(t)) \text{ where: } \begin{cases} s = j\omega \quad (\sigma = 0) \\ x(t) = 0 \text{ for all } t \leq 0 \\ \qquad \text{I.C.} \equiv 0 \end{cases}$$

If I.C. $= 0$:

$$x(s) = H(s)\cdot f(s)$$

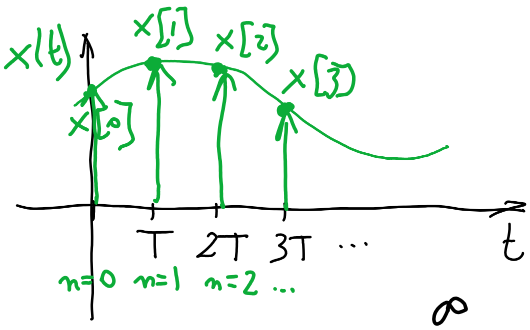
$$x(j\omega) = H(j\omega)\cdot f(j\omega)$$

Transfer function in
Laplace and Fourier domains
($s = j\omega$)

z-transform

$$\mathcal{Z}\left(x[n]\right) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

Laplace for discrete-time signals



$$x(t) \approx \sum_{n=0}^{\infty} x[n]\, \delta(t - nT)$$

$$\mathcal{L}\left(x(t)\right) \approx \sum_{n=0}^{\infty} x[n]\, \underbrace{e^{-snT}}_{\left(e^{+sT}\right)^{-n}} = \mathcal{Z}\left(x[n]\right)$$

$$z = e^{sT}$$

z represents a one-unit (*T*) *time advance* in the Laplace domain