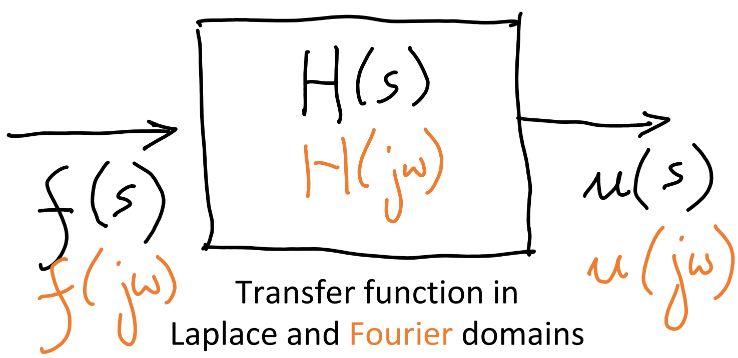
Lecture 4: System response and transfer function

Tuesday, October 13, 2020 8:36 AM

References:

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 8 (Sec. 8.2 - 8.6).

Transfer function (see Lecture 3):



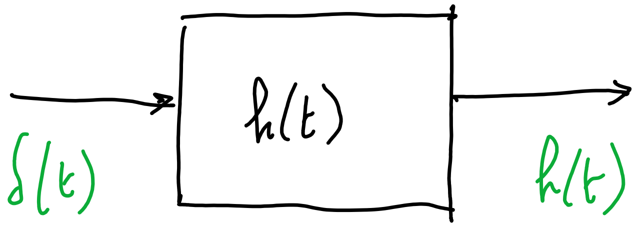
Transfer function in
Laplace and Fourier domains
($s = j\omega$)

$$\text{I.C.} = 0 \Rightarrow u\big(\underset{j\omega}{s}\big) = H\big(\underset{j\omega}{s}\big) \cdot f\big(\underset{j\omega}{s}\big)$$

For zero initial conditions (I.C.), the system response *u* to an input *f* is directly proportional to the input.

The *transfer function*, in the Laplace/Fourier domain, is the relative strength of that linear response.

Impulse response:



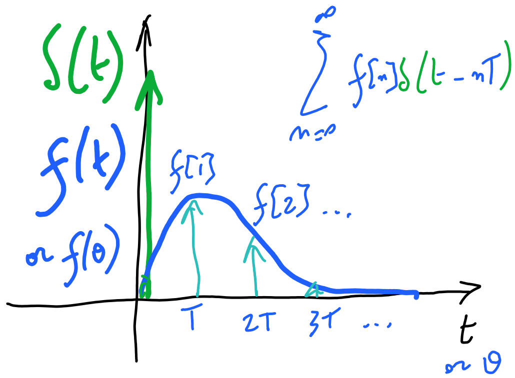
$\delta(t)$ impulse | Impulse response In the time domain | $h(t)$ impulse response

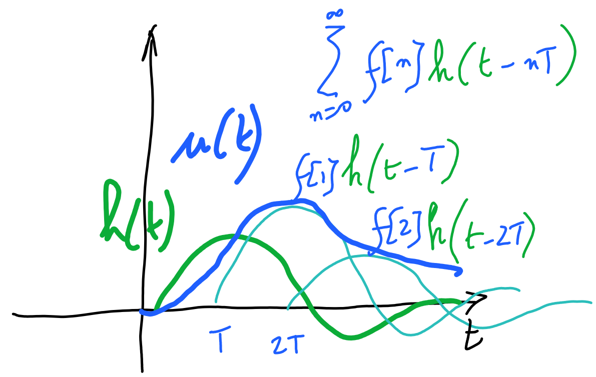
$f(t)$ input | | $u(t)$ system response

Signal decomposition into discrete-sample impulses (see Lecture 3):

$$f(t) \approx \sum_{n=0}^{\infty} f[n]\,\delta(t-nT) \qquad\qquad u(t) \approx \sum_{n=0}^{\infty} f[n]\,h(t-nT)$$

System response owing to superposition *linearity* and *time-invariance* in the impulse response





Continuous-time limit:

(sum over *n* becomes integral over θ)

$$f(t) = \int_0^{+\infty} f(\theta)\,\delta(t-\theta)\,d\theta \qquad\qquad u(t) = \int_0^{+\infty} f(\theta)\,h(t-\theta)\,d\theta$$

System response is the **convolution** of the input and the impulse response, in the time domain.

Convolution in the time domain maps to multiplication in the Laplace/Fourier domain:

$$u(s) = H(s) \cdot f(s)$$

Correspondingly, the *transfer function* is the Laplace/Fourier transform of the impulse response.

$$H(s) = \mathcal{L}(h(t))$$

$$H(j\omega) = \mathcal{F}(h(t))$$

Proof-- convolution in time maps to multiplication in the Laplace/Fourier domain:

$$\mathcal{L}(u(t)) = \mathcal{L}\left(\int_0^{+\infty} f(\theta)\, h(t-\theta)\, d\theta \right) = \mathcal{L}(f * h)$$

("*" denotes convolution)

$$= \int_0^{+\infty} \int_0^{+\infty} f(\theta)\, h(\underbrace{t-\theta}_{t'})\, d\theta \; e^{-st} \, dt$$

$$e^{-st'} e^{-s\theta} \, dt'$$

$$t = t' + \theta$$

$$= \underbrace{\int_0^{+\infty} f(\theta)\, e^{-s\theta} \, d\theta}_{\mathcal{L}(f(\theta))} \cdot \underbrace{\int_0^{+\infty} h(t')\, e^{-st'} \, dt'}_{\mathcal{L}(h(t'))}$$

$$= f(s) \cdot h(s)$$

First-order system:

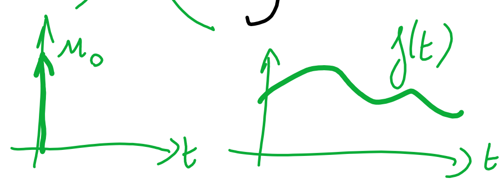
$$\frac{d\overset{(\rightarrow)}{u}}{dt} = -\underset{(=)}{\frac{1}{\tau}}\, u + f(t)$$

$$A\,\overset{(\rightarrow)}{u} + \overset{(\rightarrow)}{b}\, f(t) \qquad A = -\frac{1}{\tau}, \quad b = 1$$

$$\left[\frac{U}{s}\right] \qquad \left[\frac{F}{Us}\right]\,[F]$$

$$s\,u(s) - \underset{(I.C.)}{u_0} = -\frac{1}{\tau}\,u(s) + f(s)$$

$$\left(s + \frac{1}{\tau}\right) u(s) = \cancel{u_0} + f(s)$$

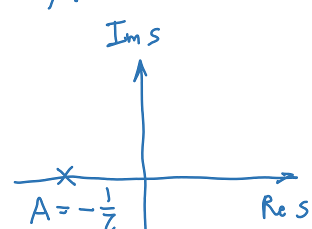


A non-zero I.C. $u(0) = u_0$ is equivalent to an additional driving force $f(t) = u_0\,\delta(t)$.

For purposes of defining the system response and transfer function, we ignore I.C.s, and consider the system were activated with a driving force $f(t)$ at all times, starting well before $t = 0$.

Transfer functions in Laplace/Fourier:

$$H(s) = \frac{u(s)}{f(s)} = \frac{1}{s + \frac{1}{\tau}} = \frac{b}{s - A} \qquad A = -\frac{1}{\tau}, \quad b = 1$$

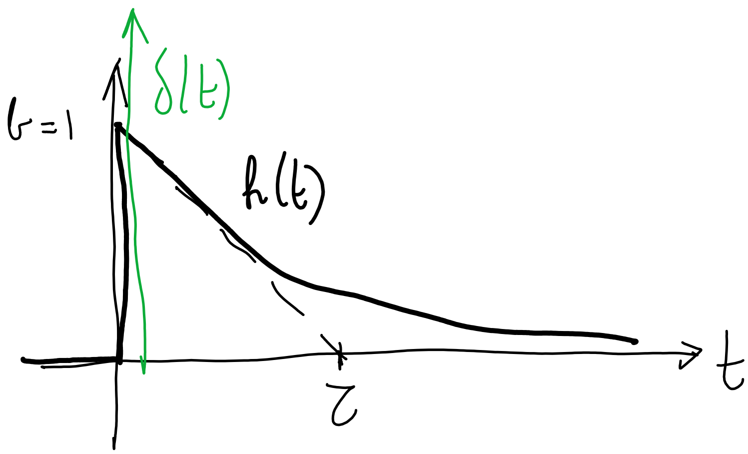


$$H(j\omega) = \frac{u(j\omega)}{f(j\omega)} = \frac{1}{j\omega + \frac{1}{\tau}}$$

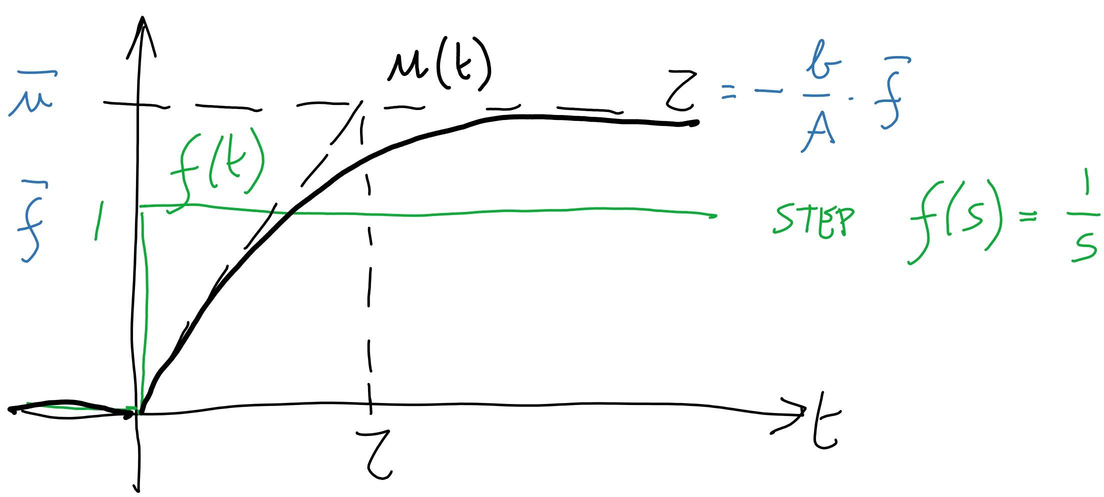
single real, negative pole

Impulse response (inverse Laplace of transfer function):

$$h(t) = e^{-t/\tau} \qquad = b\, e^{At}$$



Step response:



$$u(s) = \frac{b}{s-A} f(s) = b \frac{1}{s(s-A)} = b \frac{1}{A}\left(-\frac{1}{s} + \frac{1}{s-A}\right)$$

$$u(s) = \tau\left(\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right)$$

$$u(t) = \begin{cases} \tau\left(1 - e^{-t/\tau}\right) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

$$\text{S.S.:} \quad \bar{u} = u(\infty) = \underbrace{-\frac{b}{A}}_{H(j\omega=0)} \cdot \bar{f}$$

Steady-state

Transfer function at zero frequency (DC)

Note: step response is integral of impulse response, since $u(s) = 1/s\, h(s)$.

Second-order system:

Canonical form (2 coupled LTI ODEs in u and v):

$$\begin{cases} \dfrac{du}{dt} = v \\[2ex] \dfrac{dv}{dt} = \dfrac{f_{total}}{m} = \dfrac{1}{m}\left(-\gamma v - k u + f(t)\right) \end{cases}$$

$$\text{I.C.} = 0: \quad \begin{aligned} u_0 &= 0 \\ v_0 &= 0 \end{aligned}$$

Explicit form (Second-order ODE in u, substituting v and its derivative):

$$\mathcal{L}\left(m \frac{d^2u}{dt^2} + \gamma \frac{du}{dt} + k u = f(t) \right)$$

$$\text{I.C.} = 0: \quad \begin{aligned} u(0) &= u_0 = 0 \\ \frac{du}{dt}(0) &= v_0 = 0 \end{aligned}$$

Transfer function:

$$(m s^2 + \gamma s + k)\, u(s) = f(s)$$

$$H(s) = \frac{u(s)}{f(s)} = \frac{1}{m s^2 + \gamma s + k}$$

$$= \frac{1}{m(s-\alpha_1)(s-\alpha_2)} \qquad \alpha_{1,2} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4km}}{2m}$$

$$= \frac{1}{m(\alpha_1-\alpha_2)}\left(\frac{1}{s-\alpha_1} - \frac{1}{s-\alpha_2}\right)$$

Impulse response (inverse Laplace of transfer function):

$$h(t) = \frac{1}{m(\alpha_1-\alpha_2)}\left(e^{\alpha_1 t} - e^{\alpha_2 t}\right) \qquad \begin{aligned}\alpha_1 &= -\frac{1}{\tau_1}\\ \alpha_2 &= -\frac{1}{\tau_2}\end{aligned} \quad \text{for } \gamma > 2\sqrt{km}$$

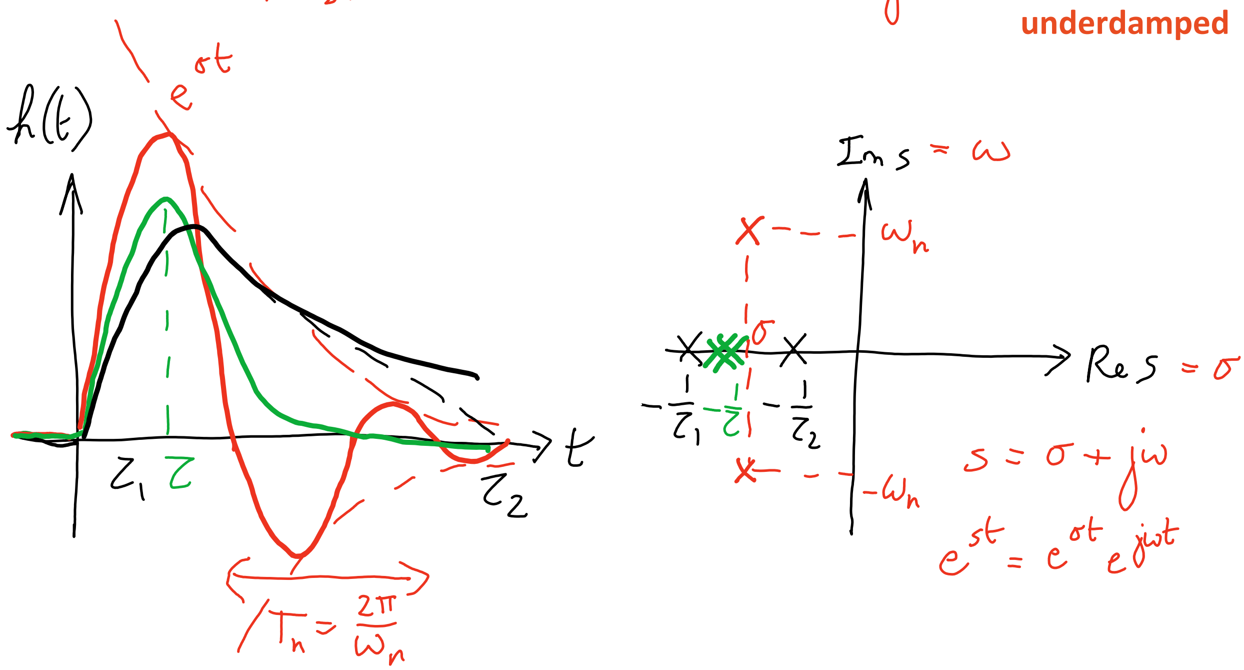
overdamped

$$= \frac{1}{m}\, t\, e^{\alpha t} \qquad \alpha = \alpha_1 = \alpha_2 \qquad \gamma = 2\sqrt{km}$$

critically damped

$$= \frac{1}{m(\alpha_1-\alpha_2)}\left(e^{\alpha_1 t} - e^{\alpha_2 t}\right) \qquad \begin{aligned}\alpha_1 &= \sigma + j\omega_n\\ \alpha_2 &= \sigma - j\omega_n\end{aligned} \qquad \gamma < 2\sqrt{km}$$

underdamped



Step response (integral of impulse response):