Lecture 4: System response and transfer function

References:


Transfer function (see Lecture 3):

\[
\begin{array}{c}
\text{Transfer function in Laplace and Fourier domains} \\
(s = j\omega)
\end{array}
\]

For zero initial conditions (I.C.), the system response \( u \) to an input \( f \) is directly proportional to the input.

The *transfer function*, in the Laplace/Fourier domain, is the relative strength of that linear response.
Impulse response:

**Signal decomposition into discrete-sample impulses (see Lecture 3):**

\[ f(t) = \sum_{n=0}^{\infty} f[n] \delta(t-nT) \]

\[ m(t) = \sum_{n=0}^{\infty} f[n] h(t-nT) \]

System response owing to superposition *linearity* and *time-invariance* in the impulse response

Continuous-time limit:
(sum over \( n \) becomes integral over \( \theta \))

\[ f(t) = \int f(\theta) \delta(t-\theta) \, d\theta \]

\[ m(t) = \int f(\theta) h(t-\theta) \, d\theta \]

System response is the **convolution** of the input and the impulse response, in the time domain.
Convolution in the time domain maps to multiplication in the Laplace/Fourier domain:

\[ m(s) = H(s) \cdot f(s) \]

Correspondingly, the transfer function is the Laplace/Fourier transform of the impulse response.

\[ H(s) = \mathcal{L}(h(t)) \]

\[ H(j\omega) = \mathcal{F}(h(t)) \]

Proof-- convolution in time maps to multiplication in the Laplace/Fourier domain:

\[
\mathcal{L}(m(t)) = \mathcal{L}\left( \int_0^{+\infty} f(\tau) h(t-\tau) \, d\tau \right) = \mathcal{L}\left( f * h \right) \\
= \int_0^{+\infty} \int_0^{+\infty} f(\tau) h(t-\tau) \, d\tau \, e^{-st} \, d\tau \\
= \int_0^{+\infty} f(\tau) e^{-st} \, d\tau \int_0^{+\infty} h(t-\tau) e^{-s(t-\tau)} \, d\tau \\
= \mathcal{L}(f(s)) \cdot \mathcal{L}(h(t')) \\
= f(s) \cdot h(s) \]
First-order system:

\[
\frac{du}{dt} = -\frac{1}{\ell} u + \int f(t) \quad A = -\frac{1}{\ell} \quad \ell = 1
\]

\[
\begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} F \\ U_s \end{bmatrix} \begin{bmatrix} F \end{bmatrix}
\]

\[
S \cdot u(s) - m_0 = -\frac{1}{\ell} u(s) + f(s) \quad \text{(I.C.)}
\]

\[
(s + \frac{1}{\ell}) \cdot u(s) = m_0 + f(s)
\]

A non-zero I.C. \( u(0) = u_0 \) is equivalent to an additional driving force \( f(t) = u_0 \delta(t) \).

For purposes of defining the system response and transfer function, we ignore I.C.s, and consider the system were activated with a driving force \( f(t) \) at all times, starting well before \( t = 0 \).

Transfer functions in Laplace/Fourier:

\[
H(s) = \frac{U(s)}{F(s)} = \frac{1}{s + \frac{1}{\ell}} = \frac{b}{s - A} \quad A = -\frac{1}{\ell} \quad \ell = 1
\]

\[
H(j\omega) = \frac{U(j\omega)}{F(j\omega)} = \frac{1}{j\omega + \frac{1}{\ell}}
\]

single real, negative pole
Impulse response (inverse Laplace of transfer function):

\[ h(t) = e^{-t/\tau} = b \ e^{At} \]

Step response:

\[ u(s) = \frac{b}{s-A} \cdot \frac{1}{s} = b \ \frac{1}{s} \left( -\frac{1}{s} + \frac{1}{s-A} \right) \]

\[ m(s) = 2 \left( \frac{1}{s} - \frac{1}{s+\frac{1}{2}} \right) \]
\[ m(t) = \begin{cases} 2 \left( 1 - e^{-4t/2} \right) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases} \]

S.S. : \[ \overline{m} = m(\infty) = -\frac{b}{A} \]

**Transfer function at zero frequency (DC):**

\[ H(j\omega = 0) \]

**Note:** step response is integral of impulse response, since \( u(s) = 1/s \, h(s) \).

**Second-order system:**

**Canonical form (2 coupled LTI ODEs in \( u \) and \( v \)):**

\[
\begin{align*}
\frac{du}{dt} &= v \\
\frac{dv}{dt} &= \frac{1}{m} \left( -\gamma v - ku + f(t) \right)
\end{align*}
\]

I.C. : \( v_0 = 0 \), \( u_0 = 0 \)

**Explicit form (Second-order ODE in \( u \), substituting \( v \) and its derivative):**

\[
\begin{align*}
L_u \left( m \frac{d^2 u}{dt^2} + \gamma \frac{du}{dt} + ku \right) &= f(t) \\
\text{I.C. :} \quad & u(0) = u_0 = 0 \\
& \frac{du}{dt}(0) = v_0 = 0
\end{align*}
\]
Transfer function:

\[
(m \ s^2 + \gamma s + k) \ u(s) = f(s)
\]

\[
H(s) = \frac{u(s)}{f(s)} = \frac{1}{m \ s^2 + \gamma s + k}
\]

\[
= \frac{1}{m(s-a_1)(s-a_2)}
\]

\[
a_1 = \frac{-\gamma}{2m} + \sqrt{\frac{\gamma^2 - 4km}{2m}}
\]

Impulse response (inverse Laplace of transfer function):

\[
h(t) = \frac{1}{m(a_1-a_2)} (e^{a_1 t} - e^{a_2 t})
\]

\[
a_1 = \frac{-\gamma}{2m}, \quad a_2 = \frac{-\gamma}{2m}
\]

\[
\text{overdamped} \quad \sigma > 2\sqrt{km}
\]

\[
\text{critically damped} \quad \gamma = 2\sqrt{km}
\]

\[
\text{underdamped} \quad \gamma < 2\sqrt{km}
\]
Step response (integral of impulse response):

\[ T_n = \frac{2\pi}{\omega_n} \]

Proof -- convolution in time maps to multiplication in the Laplace/Fourier domain:

\( * \) denotes convolution

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