Lecture 6: Stability
Tuesday, October 20, 2020  8:50 AM

References:

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 6 (Sec. 6.1 - 6.5).

Ahmed R. *MATLAB Simulink Tutorial*, Udemy.

System response $\tilde{u}$ to small-signal perturbations $\tilde{f}$ around operating point $\bar{f}$:

$$
\begin{align*}
\tilde{f} + \tilde{f}(t) &= \tilde{f}(t) \\
&\xrightarrow{\text{(LTI) ODE}} H(s) \\
&\xrightarrow{\mu(t) = \bar{\mu} + \tilde{\mu}(t)} 0 \quad \text{ideally (static)} \\
&\xrightarrow{\mu(t) \neq 0 \quad \text{transients}} \frac{\tilde{s}(t-t_0)}{\tilde{h}(t-t_0)}
\end{align*}
$$

A system is *stable* when small-signal perturbations give rise to *bounded* transients over time in the system response.

These perturbations can arise due to environmental noise, or simply due to intrinsic noise in the system, as random impulses over time. Transients can even arise in the absence of any input, due to non-zero initial conditions.

A LTI system is stable when the impulse response is bounded over time.
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A LTI system is stable when the impulse response is bounded over time.

Transfer function:

\[ H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_m s^n + a_{m-1} s^{n-1} + \ldots + a_0} \]

\[ u(s) = H(s) \cdot f(s) \]

\[ \sim \neq 0 \quad \sim = \infty \quad \sim = 0 \]

@ poles of \( H(s) \)
roots of denominator

Poles and zeros:

\[ H(s) = \frac{\lim (s-z_1)(s-z_2)\ldots(s-z_m)}{a_m (s-p_1)(s-p_2)\ldots(s-p_n)} \]

\( m \) zeros
\( n \) poles

Homework 2 questions (office hours):

- Problem 1 - Step response settling time
- Problem 2 - Linearization

Lecture 6: Stability
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8:50 AM
Complex poles always come in complex conjugate pairs: 
\[ p = \sigma + j\omega \]
\[ p^* = \sigma - j\omega \]

Because coefficients must be real:
\[
(s - p)(s - p^*) = s^2 - 2\text{Re}(p)s + |p|^2
= s^2 - 2\sigma s + \sigma^2 + \omega^2.
\]

Interpretation of complex poles in the time domain:
\[
e^{pt} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} \cos(\omega t) + j e^{\sigma t} \sin(\omega t)
\]

- exponential rise/decay with rate \(\sigma\)
- oscillation with radial frequency \(\omega\)

Small-signal transients to an input perturbation:
\[
\tilde{w}(s) = H(s) \tilde{f}(s)
\]

Partial fraction decomposition of transfer function:
\[
H(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \ldots + \frac{c_n}{s - p_n} + c_0 + \ldots
\]

- \(p_1 \neq p_2 \neq \ldots \neq p_n\)
- \(i\) if not: \(\frac{al}{(s - s_i)^2}\)

Impulse response (inverse Laplace):
\[
h(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \ldots + c_n e^{p_n t} + c_0 \delta(t) + \ldots
\]
Real poles give rise to exponentials. Any positive exponent causes instability.

Complex conjugate pairs of poles \( (p_k \text{ and } p_l = p_k^*) \) combine to give rise to cosine/sine modulated exponentials:

\[
p_k = \delta_k \pm j\omega_k:
\]

\[
C_k e^{p_k t} + C_l e^{p_l t}
\]

\[
= C_k e^{(\delta_k + j\omega_k) t} + C_l e^{(\delta_k - j\omega_k) t}
\]

\[
\text{c.c., complex conjugate}
\]

\[
= e^{\delta_k t} \left( C_k e^{j\omega_k t} + C_l e^{-j\omega_k t} \right)
\]

\[
= e^{\delta_k t} \left( \frac{C_k e^{j\omega_k t} + C_l e^{-j\omega_k t}}{2} \right) + \text{Im} C_k e^{j\omega_k t} \frac{C_k e^{j\omega_k t} - C_l e^{-j\omega_k t}}{2}
\]

\[
A_k \cos(y_k) \cdot \cos(wt) + A_k \sin(y_k) \cdot (-\sin(wt))
\]

\[
e^{jut} = \cos(wt) + j \sin(wt) \quad \cos(wt) = \frac{e^{jut} + e^{-jut}}{2}
\]

\[
e^{-jut} = \cos(wt) - j \sin(wt) \quad \cos(wt) = \frac{e^{jut} - e^{-jut}}{2j}
\]

\[
C_k = A_k e^{j\delta_k} \quad \text{Re} C_k = A_k \cos(y_k)
\]

\[
\text{Im} C_k = A_k \sin(y_k)
\]
A system is stable when small signal perturbations give rise to bounded transients over time in the system response. These perturbations can arise due to environmental noise, or simply due to intrinsic noise in the system, as random impulses over time. Transients can even arise in the absence of any input, due to non-zero initial conditions.

A LTI system is stable when the impulse response is bounded over time. Transfer function:

$$H(s) = \frac{\text{poles of } H(s)}{\text{roots of denominator}}$$

Poles and zeros:

- Complex poles always come in complex conjugate pairs:
- Because coefficients must be real:

**Interpretation of complex poles in the time domain:**

- Exponential rise/decay with rate $s$
- Oscillation with radial frequency $\omega$

**Small signal transients to an input perturbation:**

Partial fraction decomposition of transfer function:

Impulse response (inverse Laplace):

- Real poles give rise to exponentials. Any positive exponent causes instability.
- Complex conjugate pairs of poles ($p_k$ and $p_l^*$) combine to give rise to cosine/sine modulated exponentials:

Complex conjugate pairs of poles:

$$c_{e^{pt}} + c_{e^{pt^*}} = A_k e^{pt} \cos(w_k t + \phi_k)$$

Real poles:

$$c_{e^{pt}} = c_k e^{pt}$$

**Oscillations** (don't affect stability)

**Stability:**

- $\delta_k < 0$
- $\text{Re}(\phi_k) < 0$

**Critically stable:**

- $\delta_k = 0$
- $\text{Re}(\phi_k) = 0$

Strict decay in amplitude

No growth in amplitude (possibly maintaining amplitude)

Complex amplitude and phase, and Fourier system response:

LTI ODE

$$H(s) = \frac{H(j\omega)}{s}$$

$$H(j\omega) \cdot e^{j\varphi(s)} = A_k e^{j(\omega_k t + \phi_k)}$$

Homework 2 questions (office hours):

- Problem 2
- Linearization:
- Problem 1
- Step response settling time:

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