Lecture 6: Stability

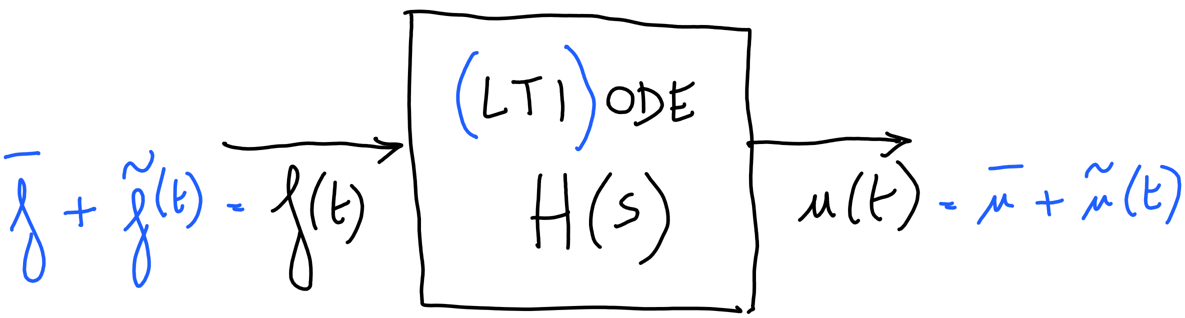
Tuesday, October 20, 2020 8:50 AM

References:

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 6 (Sec. 6.1 - 6.5).

Ahmed R. *MATLAB Simulink Tutorial*, Udemy.

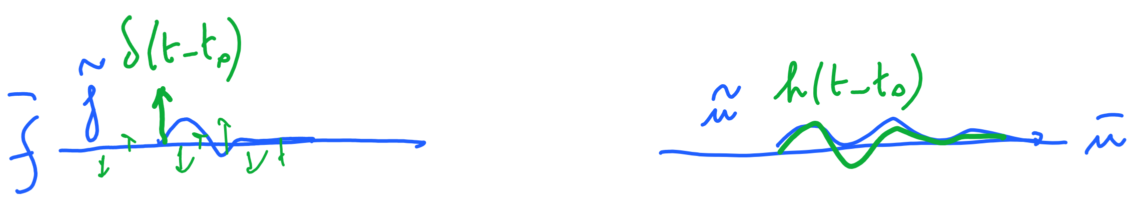
System response $\tilde{u}$ to small-signal perturbations $\tilde{f}$ around operating point $\bar{f}$:



0

0 ideally (static)

$u(t) \neq 0$ transients



A system is *stable* when small-signal perturbations give rise to *bounded* transients over time in the system response.

These perturbations can arise due to environmental noise, or simply due to intrinsic noise in the system, as random impulses over time. Transients can even arise in the absence of any input, due to non-zero initial conditions.

A LTI system is stable when the impulse response is bounded over time.

Transfer function:

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots b_0}{d_n s^n + d_{n-1} s^{n-1} + \dots + d_0}$$

$$\underbrace{\tilde{u}(s)}_{\neq 0} = \underbrace{H(s)}_{= \infty} \cdot \underbrace{\tilde{f}(s)}_{= 0}$$

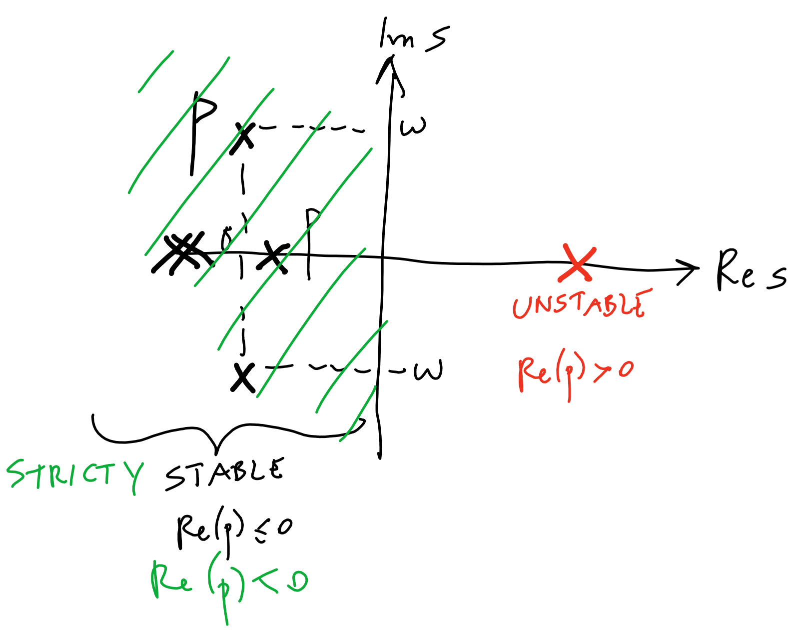
@ poles of $H(s)$
roots of denominator

Poles and zeros:

$$H(s) = \frac{b_m (s - z_1)(s - z_2) \dots (s - z_m)}{d_n (s - p_1)(s - p_2) \dots (s - p_n)}$$

m zeros

n poles



Complex poles always come in complex conjugate pairs:

$$\begin{cases} p = \sigma + j\omega \\ p^* = \sigma - j\omega \end{cases}$$

Because coefficients must be real:

$$(s-p)(s-p^*) = s^2 - 2\,\mathrm{Re}(p)\,s + |p|^2$$
$$= s^2 - 2\sigma s + \sigma^2 + \omega^2$$

Interpretation of complex poles in the time domain:

$$e^{pt} = e^{\sigma t} \quad e^{j\omega t}$$
$$e^{p^* t} = \underbrace{e^{\sigma t}}_{e^{\sigma t}} \quad \underbrace{e^{-j\omega t}}_{\substack{\cos(\omega t) \\ \sin(\omega t)}}$$

exponential rise/decay with rate σ

oscillation with radial frequency ω

Small-signal transients to an input perturbation:

$$\tilde{w}(s) = H(s) \quad \tilde{f}(s)$$

Partial fraction decomposition of transfer function:

$$H(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \dots + \frac{c_n}{s-p_n} + c_0 + \dots$$

$p_1 \neq p_2 \neq \dots \neq p_n$ if not: $\frac{d_i}{(s-s_i)^2}$

Impulse response (inverse Laplace):

$$h(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \dots c_n e^{p_n t} + c_0 \delta(t) + \dots$$

Real poles give rise to exponentials. Any positive exponent causes instability.

Complex conjugate pairs of poles (p_k and $p_l = p_k^*$) combine to give rise to cosine/sine modulated exponentials:

$$p_{k,l} = \sigma_k \pm j\omega_k :$$

$$c_k e^{p_k t} + c_l e^{p_l t}$$

$$= c_k e^{\overbrace{(\sigma_k + j\omega_k)}^{p_k} t} + \underbrace{\underbrace{c_k^*}_{c_l} e^{\overbrace{(\sigma_k - j\omega_k)}^{p_l = p_k^*} t}}_{\text{c.c.}}$$

complex conjugate

$$= e^{\sigma_k t} \left(\underset{(Re + j\,Im)}{c_k} e^{j\omega_k t} + \underset{(Re - j\,Im)}{c_k^*} e^{-j\omega_k t} \right)$$

$$c_k = \mathrm{Re}\, c_k + j\, \mathrm{Im}\, c_k$$

$$= e^{\sigma_k t} \left(\underbrace{\mathrm{Re}\, c_k}_{A_k \cos(\varphi_k)} \cdot \underbrace{\frac{e^{j\omega_k t} + e^{-j\omega_k t}}{2}}_{\cos(\omega_k t)} + \underbrace{\mathrm{Im}\, c_k}_{A_k \sin(\varphi_k)} \cdot \underbrace{j \frac{e^{j\omega_k t} - e^{-j\omega_k t}}{2}}_{(-\sin(\omega_k t))} \right)$$

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \qquad \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t)$$

$$= \cos(-\omega t) + j \sin(-\omega t) \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$c_k = A_k e^{j\varphi_k} \qquad \mathrm{Re}\, c_k = A_k \cos \varphi_k$$

$$\mathrm{Im}\, c_k = A_k \sin \varphi_k$$

Complex conjugate pairs of poles:

$$c_k e^{p_k t} + c_k^* e^{p_k^* t} = A_k e^{\sigma_k t} \cos(\omega_k t + \varphi_k)$$

Oscillations
(don't affect stability)

Real poles:

$$c_k e^{p_k t} = c_k e^{p_k t}$$

Stability:

$\sigma_k < 0$

$\mathrm{Re}(p_k) < 0$

$\downarrow$

Strict decay in amplitude

Critically stable:

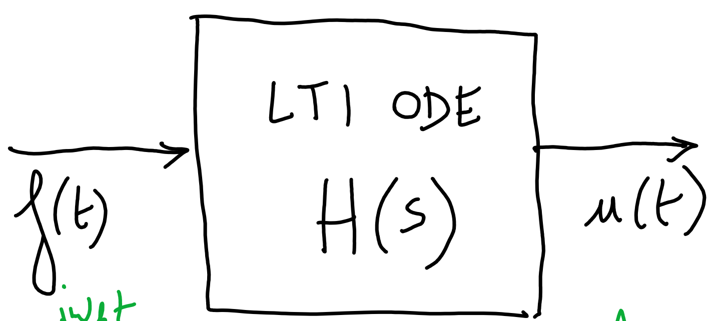
$\sigma_k \leq 0$

$\mathrm{Re}(p_k) \leq 0$

$\downarrow$

No growth in amplitude
(possibly maintaining amplitude)

Complex amplitude and phase, and Fourier system response:



$e^{j\omega_k t}$

$A_k e^{j(\omega_k t + \varphi_k)}$

$$H(j\omega) = A(s) e^{j\varphi(s)}$$