Lecture 9: Proportional, Integral, Derivative Control

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References:

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 9 (Sec. 9.4 - 9.5).

General control setting (Lecture 8):

![Control System Diagram]

Our task is to design the controller $F(s)$, based on a model of the biosystem $H(s)$, and the measurement feedback system $G(s)$.
We consider a mixture of three control strategies for $F(s)$.

Proportional (P) control:

$$F = K_p$$

$$f(t) = K_p \cdot e(t) = K_p \left( u_{\text{TARGET}} - u_{\text{MEAS}} \right)$$

The effect of this design choice for controller $F(s)$ on the closed-loop dynamics $CL(s)$ depends on the models for the biosystem $H(s)$ and measurement $G(s)$.

Consider ideal measurement:

$$G(s) = 1$$

and two cases of an example biosystem with second-order dynamics:

$$H(s) = \frac{1}{m s^2 + \gamma s + k}$$

e.g., position of force-driven damped spring-mass system (second-order lowpass)

$$H(s) = \frac{s}{m s^2 + \gamma s + k}$$

e.g., velocity of force-driven damped spring-mass system (second-order bandpass)
Resulting closed-loop transfer function, in both cases:

\[
CL(s) = \frac{s F(s)}{m s^2 + \gamma s + \kappa} = \frac{s F(s)}{m s^2 + \gamma s + \kappa + s F(s)}
\]

which for proportional control reduce to:

\[
CL(s) = \frac{K_p s}{m s^2 + \gamma s + \kappa + K_p s}
\]

\(K_p\) increases speed of the low-pass response, but with steady-state error, and ringing (under-damped oscillations).

\(K_p\) increases damping of the low-pass response, but without steady-state response, and with slow (over-damped) settling.

Steady-state gain \((s = 0)\):

\[
\frac{K_p}{\kappa + K_p} < 1
\]

These steady-state errors are remediated by adding integral control.

The dynamics are further improved by adding derivative control.
Proportional and integral (PI) control:

\[ F(s) = K_p + K_i \frac{1}{s} = \frac{K_p s + K_i}{s} \]

\[ f(t) = K_p e(t) + K_i \int_{-\infty}^{t} e(t) \, dt \]

\[ CL(s) = \frac{sF(s)}{ms^2 + \gamma s + k + sF(s)} = \frac{K_p s + K_i}{ms^2 + \gamma s + k + \frac{K_p s + K_i}{s}} \]

\[ = \frac{K_p s + K_i}{ms^3 + \gamma s^2 + (k + K_p) s + K_i} \]

Steady-state gain \( (s = 0) \):

\[ = 1 \]

Adding derivative control further improves on the high-frequency response.
Proportional, integral and derivative (PID) control:

\[
F(s) = K_p + K_i \frac{1}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}
\]

\[
f(t) = K_p e(t) + K_i \int_{-\infty}^{t} e(t) \, dt + K_d \frac{d}{dt} e(t)
\]

\[
CL(s) = \frac{s F(s)}{m s^2 + \gamma s + k + s F(s)} = \frac{K_d s^2 + K_p s + K_i}{s}
\]

\[
= \frac{K_d s^2 + K_p s + K_i}{m s^3 + (\gamma + K_d) s^2 + (k + K_p) s + K_i}
\]

at low frequencies (\(s = 0\)): \(L(s) \approx 1\) (\(K_i \neq 0\)) \quad at low frequencies (\(s = 0\)): \(\frac{K_i}{k + K_i} < 1\)

at high frequencies (\(s \to \infty\)): \(\frac{K_d}{m s} \quad \text{at high frequencies (}s \to \infty\): \(\frac{K_i}{m + K_d} < 1\)

Derivative control \(K_d\) improves high-frequency response (lowers the rise time)

Integral control \(K_i\) improves low-frequency response (reduces steady-state error)

Proportional control \(K_p\) allows to improve mid-frequency response (improve settling by critically damping the response)