Practice Quiz 1 Solutions

Quiz 1 covers all material, as covered in Lectures 1 through 7, and Homework 1 through 3. It is open book, open notes, and online, but web search is prohibited. ***No collaboration or communication in any form is allowed***, except for questions to the instructor and TAs.

Quiz 1 will be posted online, and is due over Canvas as scheduled. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.

References

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 1 - Ch. 8.

PRACTICE QUIZ SOLUTIONS

Problem 1

Consider the following biochemical reaction taking place in an organ in the body:

$$A \underset{k_r}{\overset{k_d}{\rightleftharpoons}} B + C$$

where compound A decomposes into compounds B and C at rate k_d, and B and C recombine into A at rate k_r. Assume B and C are maintained at constant concentrations $[B]_0$ and $[C]_0$ inside the organ.

1. Write the ODE that describes the kinetics of the reaction in the concentration [A].

$$\frac{d}{dt}[A] = -k_d[A] + k_r[B]_0[C]_0$$

2. Find the equilibrium (*i.e.*, steady-state) concentration $[A]_0$ of compound A in the organ, and find the time constant of the reaction.

$$[A]_{s.s.} = \frac{k_r}{k_d}[B]_0[C]_0$$

$$\tau = \frac{1}{k_d}$$

3. Now, assume compound A flows out of the volume V of the organ at a flow rate Q. Write the ODE in the concentration [A] that describes both the reaction kinetics and the flow of A. Find the time constant.

$$\frac{d}{dt}[A] = -\frac{Q}{V}[A] - k_d[A] + k_r[B]_0[C]_0$$

$$\tau = \frac{1}{k_d + \frac{Q}{V}}$$

4. Use Laplace transforms to find the concentration [A] as a function of time, starting from zero initial condition $[A](0) = 0$, with the following parameters: $k_d = 0.5$ / min, $k_r = 0.1$ L / min mmol, $Q = 2$ L/min, $V = 4$ L, $[B]_0 = 10$ mmol/L, and $[C]_0 = 1$ mmol/L.

$$s\,[A](s) = -\underbrace{\left(k_d + \frac{Q}{V}\right)}_{\frac{1}{\tau}}[A](s) + \frac{1}{s}k_r[B]_0[C]_0$$

$$\left(s + \frac{1}{\tau}\right)[A](s) = \frac{1}{s}k_r[B]_0[C]_0$$

$$[A](s) = k_r[B]_0[C]_0 \,\frac{1}{s}\,\frac{1}{s+\frac{1}{\tau}}$$

$$\frac{1}{s}\,\frac{1}{s+\alpha} = \frac{1}{\alpha}\left(\frac{1}{s} - \frac{1}{s+\alpha}\right)$$

$$[A](s) = k_r[B]_0[C]_0\,\tau\left(\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right)$$

$$[A](t) = k_r[B]_0[C]_0\,\tau\left(1 - e^{-\frac{t}{\tau}}\right)$$

for $t \geqslant 0$

$\tau = 1$ min

$k_r[B]_0[C]_0\,\tau = 1$ mmol/L

$$[A](t) = 1\ \text{mmol/L}\left(1 - e^{-\frac{t}{1\ \text{min}}}\right)$$

Problem 2

Consider the following set of ODEs describing the dynamics of a biomechanical system with mass *m* and stiffness *k*, with force *f*(*t*) driving the input, and with velocity *v*(*t*) at the output:

$$\frac{du}{dt} = v(t)$$

$$m\frac{dv}{dt} = -k\,u(t) + f(t)$$

1. Find the Laplace transform of velocity *v*(*s*) as a function of the Laplace transform of the force *f*(*s*), and initial conditions in velocity $v(0) = v_0$ and in position $u(0) = u_0$.

$$s\,u(s) - u_0 = v(s)$$

$$\Big(m\big(s\,v(s) - v_0\big) = -k\,u(s) + f(s) \Big) \times s$$

$$m s^2 v(s) - m s v_0 = -k\big(v(s) + u_0\big) + s\,f(s)$$

$$\left(s^2 + \frac{k}{m}\right) v(s) = s v_0 - \frac{k}{m}u_0 + \frac{s}{m}f(s)$$

$$v(s) = \frac{s v_0 - \frac{k}{m}u_0}{s^2 + \frac{k}{m}} + \frac{\frac{s}{m}f(s)}{s^2 + \frac{k}{m}}$$

2. For zero force $f(t) = 0$, and for given initial conditions $v(0) = v_0$ and $u(0) = u_0$, find the velocity *v*(*t*) as a function of time.

$$\frac{s}{s^2+a^2} \longrightarrow \frac{d}{dt}\left(\frac{1}{a}\sin(at)\right) = \cos(at)$$

$$\frac{c}{s^2+a^2} = \frac{c}{2ja}\left(\frac{1}{s-ja} - \frac{1}{s+ja}\right) \longrightarrow \frac{c}{2ja}\left(e^{+jat} - e^{-jat}\right) = \underbrace{\frac{c}{a}}_{-a}\sin(at)$$

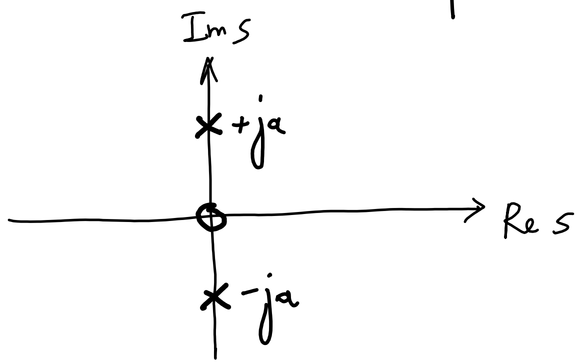
$$a = \sqrt{\frac{k}{m}}$$

$$c = -a^2$$

$$\Rightarrow v(t) = v_0 \cos(at) - \sqrt{\frac{k}{m}}\, u_0 \sin(at)$$

3. Find the transfer function $H(s) = v(s) / f(s)$ of the system, and find the poles and zeros.

$$H(s) = \frac{1}{m}\,\frac{s}{s^2+\frac{k}{m}} \qquad \begin{array}{l} \longrightarrow \text{zero @ } s = 0 \\ \longrightarrow \text{poles @ } s = \pm ja = \pm j\sqrt{\frac{k}{m}} \end{array}$$

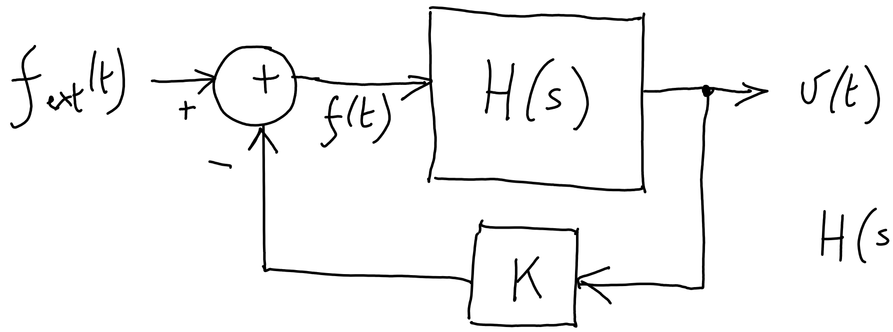


critically stable

4. Now consider closed-loop feedback, in which the force f(t) is given by

$$f(t) = f_{ext}(t) - K\,v(t)$$

where f_{ext}(t) is the externally applied force, and K is the feedback gain. Draw the closed-loop system block diagram, and find the closed-loop transfer function $F(s) = v(s) / f_{ext}(s)$.



$$H(s) = \frac{1}{m} \frac{s}{s^2 + \frac{k}{m}}$$

$$v(s) = H(s) \cdot \left(f_{ext}(s) - K\, v(s) \right)$$

$$F(s) = \frac{v(s)}{f_{ext}(s)} = \frac{H(s)}{1 + K\, H(s)}$$

5. Find the value of the feedback gain *K* for which the closed-loop system is critically damped.

Poles: $1 + K\, H(s) = 0$

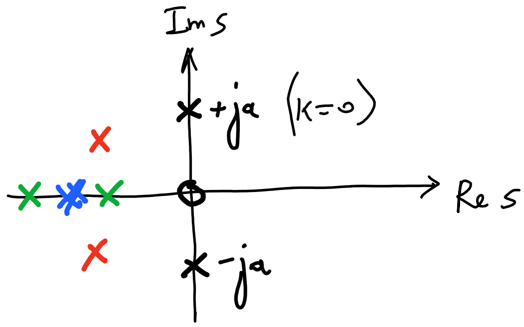
$$1 + K \frac{1}{m} \frac{s}{s^2 + \frac{k}{m}} = 0$$

$$s^2 + \frac{K}{m} s + \frac{k}{m} = 0$$

$$m s^2 + K s + k = 0$$

$$p_{1/2} = -\frac{K}{2m} \pm \frac{\sqrt{K^2 - 4km}}{2m}$$

$$D = K^2 - 4km$$



$D > 0 \Rightarrow p_{1,2}$ real **O.D.**

$D = 0 \Rightarrow p_{1,2}$ real & identical **C.D.**

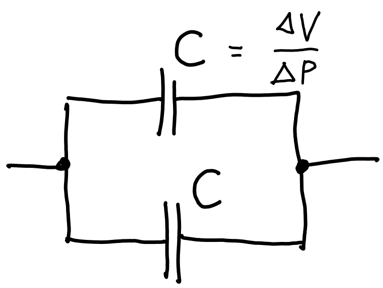
$D < 0 \Rightarrow p_{1,2}$ complex C.C. **U.D.**

C.D. @ $K^2 = 4km$ or $K = 2\sqrt{km}$

Problem 3

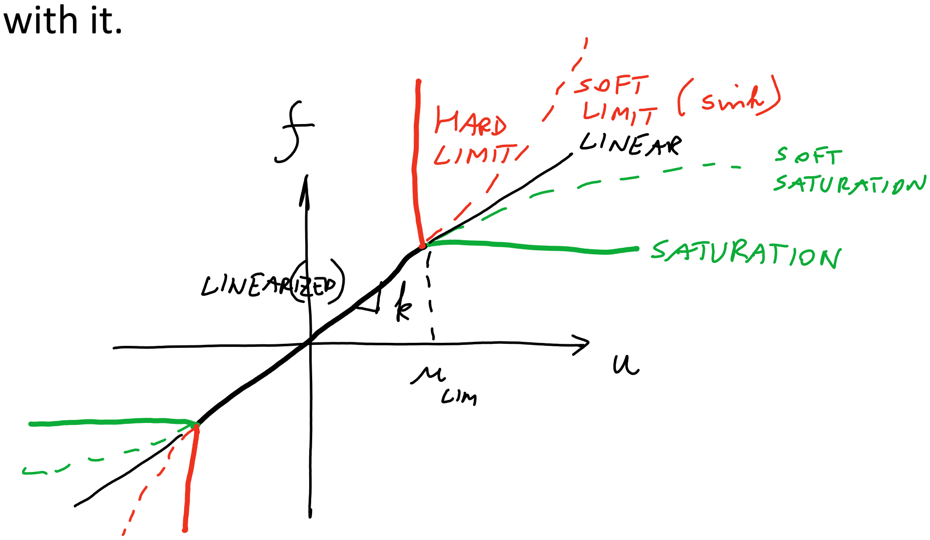
1. What is the resistance of a series junction of two vessels, each with resistance *R*? What is the compliance of a parallel junction of two vessels, each with compliance *C*?

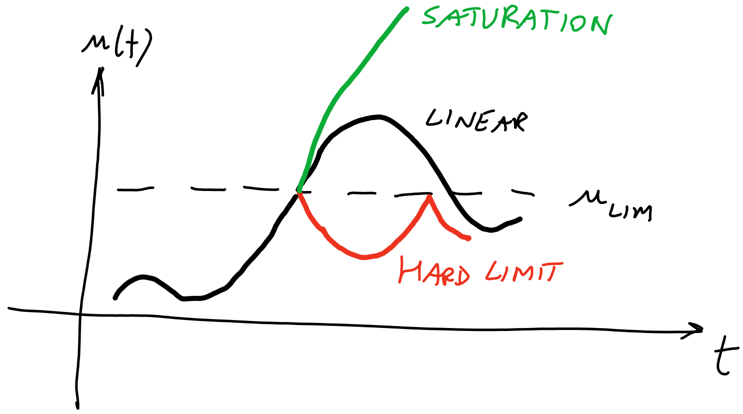
$\Rightarrow R_{total} = 2R$



$\Rightarrow C_{total} = 2C$

2. Give some example of a nonlinearity in a bioengineering system, and (briefly) describe how it affects its operation or characteristics, and how to deal with it.





3. Explain differences between the Laplace transform and the Fourier transform.

1) $\mathcal{L}: \quad 0 \leq t \leq +\infty \qquad + \text{I.C. @ } 0$

$\mathcal{F}: \quad -\infty \leq t \leq +\infty \qquad \text{No I.C.}$

2) $H(s) = H(jw)$ where $s = jw$

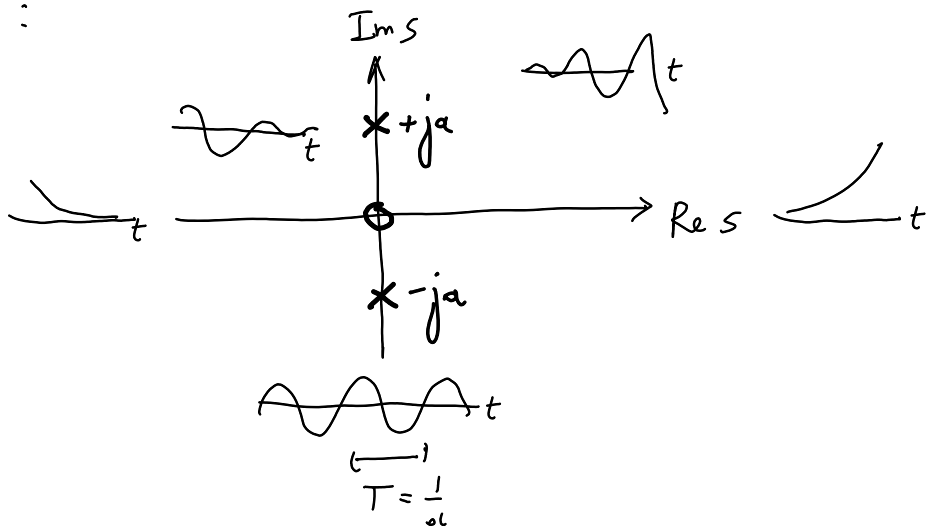
$\mathcal{L}: \quad s = \sigma + jw$ COMPLEX $\rightarrow$ exponentials

$\mathcal{F}: \quad s = jw$ PURELY IMAGINARY $\rightarrow$ sines/cosines

$\mathcal{L}: \quad u(t) = e^{\sigma t} e^{jwt} + c.c.$

$\mathcal{F}: \quad u(t) = \underbrace{e^{jwt}}_{\cos(wt) + j\sin(wt)} + c.c.$

L:



F: