2 and 3-Element Windkessel Model Analysis: The Arterial System and Its Response to Vasoactive Drug Impact

Sophia Benkirane, Sophia Lam, Angela Liu, Ly Nguyen, Nikhita Sujan
Department of Bioengineering, University of San Diego

Abstract **- By mirroring an electrical circuit to reflect parameters such as arterial compliance, peripheral resistance, and inertia, the Windkessel model (WM) displays the dynamics of the circulatory system, specifically the flow of blood through the arteries. Implementing a 2 element model and a 3 element model of the blood flow biosystem, differential equations are developed that determine the relationship between blood pressure and blood flow in the aorta as well as the compliance of blood vessels while accounting for constricted and dilated blood vessels. Further, the Laplace transform of these equations establishes a transfer function that identifies the input blood flow and output pressure response rates that were computationally analyzed through Matlab and Simulink softwares. Not only does the Windkessel model simulate dynamics of the circulatory system but it also evaluates vasodilator and vasoconstrictor drugs.**

I. INTRODUCTION

The circulatory system is a biosystem that governs the flow of blood through the aorta, ventricles, and arteries. Most importantly, it includes the cardiac cycle that is described as a closed loop system, controlled by rhythmic contractions, in which the heart pumps blood throughout the rest of the body. When the blood begins to flow into the ventricles, the aortic pressure is the lowest resulting in diastolic pressure. In the phase ventricular diastole, relaxed ventricles allow for the blood to be filled with oxygenated blood [9]. Next, ventricles contract to move the blood from the heart into the arteries in the process called systole. During the systolic phase, arterial pressure is observed to be the highest which acts as a force exerted on arterial walls. Hence, the Windkessel model demonstrates the load on the heart during the cardiac cycle and provides information that helps regulate blood flow to ensure homeostasis of the cardiovascular system.

In 1899, a German physiologist named Otto Frank designed the Windkessel model that presented the mechanics of the heart and circulatory system as a closed hydraulic circuit that comprises a water pump connected to a chamber [9]. When pumping out the water, it compresses the air in the closed compartment which pushes the water outwards as depicted in Figure 1.

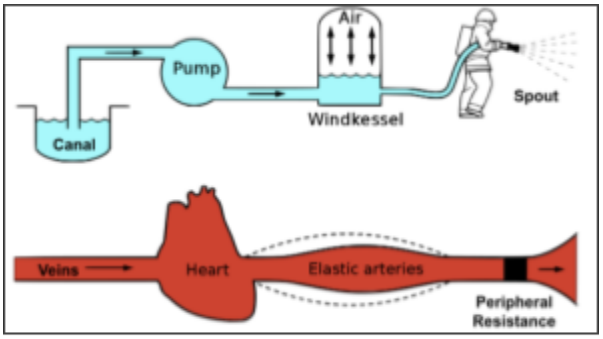

Figure 1: The Windkessel analogy illustrated [9].

The pressure changes induced through fluid dynamics is modeled by an analogous electrical circuit with voltage drops. In addition, the pressure changes are induced through systolic and diastolic phases of the cardiac cycle. Through this demonstration, equations governing the Windkessel model compare blood pressure, blood flow in the aorta, arterial compliance, peripheral resistance, and inertia. Arterial compliance, measured by the amount of blood that is pumped into the heart depending on pressure, refers to the elasticity of arteries utilized in the cardiac cycle. Peripheral resistance of the valves exhibits resistance encountered by the blood as it flows from the major arteries to capillaries. Inertia of the blood flow refers to the tendency of blood to resist change in motion as it moves through the heart during the cardiac cycle [2]. Depending on computational complexity and the curves generated for the input volumetric blood flow and output blood pressure, the Windkessel model demonstrates that when compliance occurs, aortic valves and peripheral resistance come together, insinuating that peripheral flow becomes constant. Further, the model helps regulate blood flow and maintain homeostasis in the cardiovascular system.

Vascular dynamics discusses the physiological and quantitative properties of blood vessels in relation to pressure, flow, and other factors. When blood vessels

widen due to the relaxation of the muscles in the blood vessel walls, a decrease in vascular resistance occurs as well as the lowering of blood pressure. In this scenario, the reduction in stiffness and vessel dilation enhances arterial compliance [3]. Additionally, characteristic resistance or impedance decreases as there is less resistance to the alternating blood flow in large arteries. On the other hand, vasoconstrictor drugs narrow or constrict blood vessels which increases vascular resistance, impedance, and stiffness. Higher blood pressure is observed with vasoconstrictors as there is a reduction in blood flow resulting from the reduced arterial compliance. The difference between normal, dilated, and constricted blood vessels is depicted in Figure 2 below.

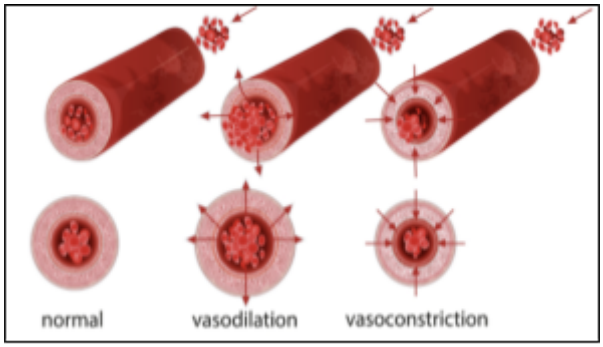

Figure 2: Illustration of Dynamic Vascular Responses: Normal, dilated, and constricted condition [8].

I. METHODS

In this project, we aim to mathematically model the aortic blood flow, blood pressure and flow relationship, and compliance of the blood vessels. These analyses will provide us with results of output blood pressure, demonstrating the robustness of the Windkessel Model for cardiac modeling and response to vasoactive drug impact. The administration of vasoactive drugs, such as vasodilators or vasoconstrictors, is common in clinical settings [3]. However, predicting their effects on the cardiovascular system and optimizing dose is challenging. The WM provides a solution by allowing simulation and analysis of the dynamic response of the cardiovascular system to these drugs.

Proposed Solution - The circulatory system is a complex system which can evaluate blood flow behavior by understanding and detailing intra-cellular interactions as well as non-linear approximations. By using lumped parameters to summarize the biosystem's behavior, an analysis of the major components can be created. To determine the relationship between blood pressure, blood flow, and compliance of blood vessels, the Windkessel model is utilized. Hence, the goal is to understand and simulate the dynamics of blood flow and pressure via a 2 element and 3 element Windkessel model, specifically for constricted, dilated, and normal conditions, to demonstrate the influence of vasodilator and vasoconstrictor drugs. Along with creating a mathematical model of the blood flow and computing the variables, the accuracy of the Windkessel model will be determined by evaluating parameters such as cardiac output and blood pressure. In addition, this method will predict the changes in the cardiovascular system when one of the parameters including arterial compliance, peripheral resistance, and inertia are changed.

Assumptions **-** The circulatory biosystem must be assumed to be in steady state. In addition, the cardiac cycle starts at systole or the phase where the heart pumps blood from its chamber to the arteries. The period of the systole is considered 2/5th of the period of the entire cardiac cycle. Since arteries are treated as passive channels, their valve-like effects are neglected. To analyze the blood flow behavior, continuous pulsatile blood flow must be present. In the differential equations associated with the 2 element Windkessel model, the total arterial compliance is represented as a capacitor and the total peripheral resistance is represented as a resistor in the equivalent electrical circuit model. In the differential equations associated with the 3 element Windkessel model, inertia is represented by an inductor.

2 Element Windkessel Model **-** The 2 element Windkessel model accounts for arterial compliance and peripheral resistance when determining the relationship between blood pressure, blood flow, and other variables that describe the dynamics of blood flow. With the arterial compliance (C in $cm^3/mmHg$) depicted as a capacitor and the peripheral resistance (R in $mmHg * s/cm^3$) represented as a resistor, the flow ($I(t)$ in cm^3/s) of blood from the heart will be analogous to the current flowing throughout the circuit. For the 2-element model, we can assume that air pressure and air volume in the Windkessel to the heart is constant, following Poiseuille's law. Thus, the differential equations and transfer functions, derived from the Laplace Transform, associated with the 2 element Windkessel model are presented in equation (1) and (2).

$$i(t) = \frac{u(t)}{R} + C \cdot \frac{du(t)}{dt} \quad (1)$$

In which *i(t)* is the blood flow into the arterial system at time *t*, *u(t)* is the output blood pressure, *R* is the total peripheral resistance, *C* is the arterial compliance, and *du(t)/dt* is the rate of change of blood pressure concerning time.

$$I(s) = \frac{U(s)}{R} + C \cdot sU(s)$$

$$I(s) = U(s)(\frac{1}{R} + sC)$$

$$H(s) = \frac{U(s)}{I(s)} = \frac{R}{sRC+1} \quad (2)$$

In which *H(s)* is the transfer function relating the Laplace transforms of the output voltage *U(s)* to the input current *I(s)* in the frequency domain. The transfer function describes the relationship between the input current and the output voltage in the 2-element WM circuit. More specifically, it is in the first order concerning time. The presence of *sRC* in the denominator represents the time constant while the *+1* term accounts for the DC gain.

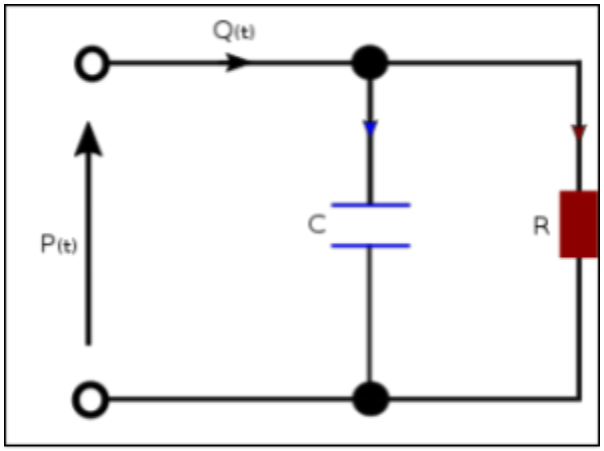

Figure 3: Two-Element electrical circuit representing cardiovascular blood flow dynamics [9].

Figure 3 demonstrates that during diastole, there is an input flow of blood, u(t), from the ventricles to the compliant aortic chamber. Here, the blood within the peripheral vessels is represented by the solid circuit lines whereas the elasticity of the aorta during diastole is depicted as the dashed capacitor. The parallel arrangement of the resistor and capacitor determines the values of the given variables by utilizing equation (1). From the electrical analog and differential equations, we were able to derive the Simulink scheme of the 2 WM as shown in Figure 4.

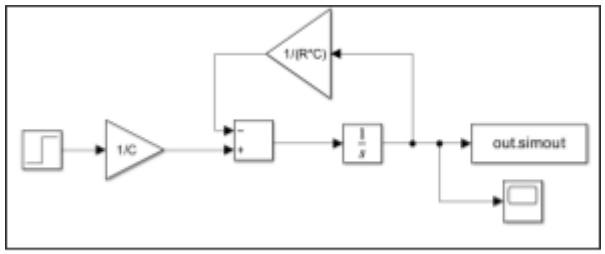

Figure 4: Two-Element WM Simulink Diagram.

Additionally, the blood pressure ($P(t)$ in $mmHg$) in the aorta is modeled as a time-varying electric potential. Throughout the simulations of each of these models, human parameters detailed in *Table 1* were utilized. These values demonstrate the effects of vasoactive drugs such as vasodilators and vasoconstrictors. More specifically, vasodilators typically relax blood vessels, leading to a reduction in resistance. Likewise, arterial compliance is increased as vessel dilation and reduced arterial stiffness are prompted. Vasoconstrictive drugs, on the other hand, lead to the narrowing of vessel walls and thus, an increase in resistance. In return, reduced arterial compliance is induced.

	R [mmHg.s.cm^{-3}]	C [cm^3.mmHg^{-1}.s^2.cm^{-3}]
Normal	0.79	1.75
Dilated	0.63	5.16
Constricted	1	1

Table 1: 2 WM Values for normal, dilated, and constricted vessel parameters [4].

3-Element Windkessel Model **-** The 3-element Windkessel model resembles a circuit, with two resistors in series that represent the arterial resistance and an additional valve resistance along with a capacitor for the arterial compliance which provides a more detailed and physiologically accurate simulation of cardiovascular dynamics. Here, it takes into account the characteristic impedance of the proximal aorta which refers to the resistance to alternating blood flow due to pressure variations that are demonstrated by volume displacements in the Windkessel setup. Partial opening and closing of the valve creates the pressure changes that impact resistance and other variables as demonstrated by

equations (3) and (4). Hence, the 3-element model related to blood pressure, blood flow, peripheral resistance, total arterial compliance, and inertance to describe the dynamics of blood flow. The differential equations and transfer functions, derived from the Laplace Transform, associated with the 3-element Windkessel model are presented in equation (3), (4), and (5).

$$i(t) = \frac{u_c(t)}{R} + C \cdot \frac{du_c(t)}{dt} \qquad (3)$$

In which *i(t)* is the total blood flow input, $u_c(t)$ is the pressure in the compliant vessel, and $du_c(t)/dt$ is the rate of change of pressure with respect to time.

$$u(t) = i(t)Z_C + u_c(t) \qquad (4)$$

In which Z_C *d*enotes the impedance of the compliant vessel.

$$U(s) = I(s)Z + U_c(s)$$

$$U_c(s) = U(s) + I(s)Z$$

$$I(s) = \frac{U_c(s)}{R} + C \cdot sU_c(s)$$

$$I(s) = U_c(s)(\frac{1}{R} + sC)$$

$$I(s) = [U(s) - I(s)Z_c] \cdot (\frac{1}{R} + sC)$$

$$I(s) = \frac{U(s)}{R} + C \cdot sU(s) - \frac{I(s)Z_c}{R} - sZ_cI(s)$$

$$H(s) = \frac{U(s)}{I(s)} = \frac{sRCZ_c + Z_c + R}{sRC + 1} \qquad (5)$$

This transfer function expresses the relationship between the Laplace-transformed voltage and current, considering the inertance (Z_C), resistance (R), and compliance (C) components in the 3 WM. The electrical analog of the 3 WM demonstrates how the arterial system responds to changes in frequency and helps further analyze its behavior in the frequency domain given these circuit elements, as seen in Figure 5.

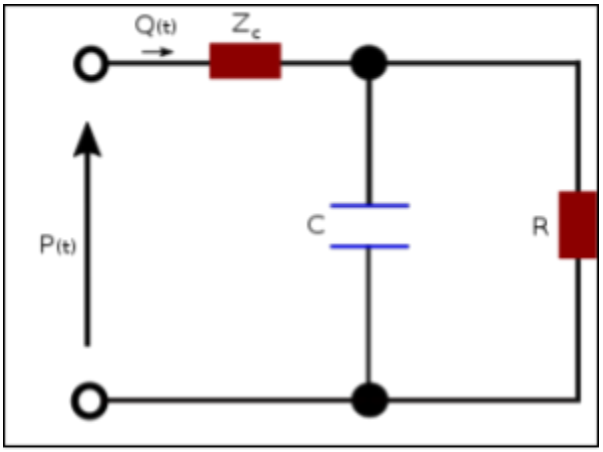

Figure 5: Three-Element electrical circuit representing cardiovascular blood flow dynamics [9].

Likewise, using the electrical analog and differential equations, we were able to derive the Simulink scheme of the 3 WM as shown in Figure 6.

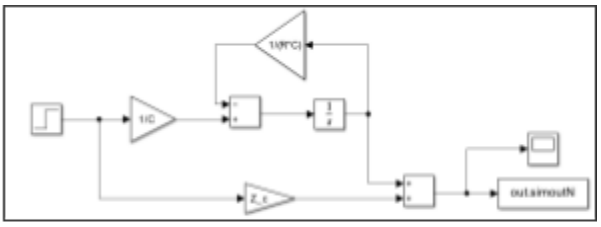

Figure 6: Three-Element WM Simulink Diagram.

Throughout the simulations of each of these models, human parameters detailed in *Table 2* were utilized. While the same vascular dynamics apply to the 3 WM, the added characteristic resistance (Z_C) demonstrates some noticeable differences between the influence of the vasoactive drugs. In such a model, vasodilators contribute to a decrease in characteristic resistance of blood flow while vasoconstricting drugs contribute to an increase.

	R [mmHg.s.cm^{-3}]	C [cm^3.mmHg^{-1}.s^2.cm^{-3}]	Z_C [mmHg.s.cm^{-3}]
Normal	0.79	1.75	0.033
Dilated	0.63	5.16	0.030
Constricted	1	1	0.050

Table 2: 3 WM Values for normal, dilated, and constricted vessel parameters [4].

II. RESULTS

Using the 2-element Simulink block diagram a step response is produced from the system's output (Figure 7). In normal conditions, the step response behavior exhibits a quick rise in blood pressure due to the sudden increase in blood flow rate in the arterial system. For dilated blood vessels, for the same blood flow rate, a faster rise in blood pressure was observed. While a lower equilibrium value for blood pressure was observed in comparison to normal conditions. For a constricted system, behavior of a slower rise in blood pressure was exhibited for the same blood flow rate input as normal and dilated conditions. However, the blood pressure reached a higher equilibrium.

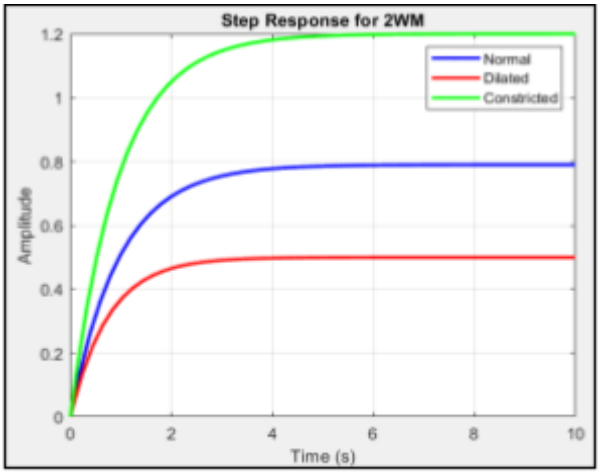

Figure 7: Normal, dilated, and constricted conditions for Step Response of 2 WM.

With a 3-element Simulink block diagram, characteristic resistance (Z_C) was incorporated into the model. The same parameters were used for the 3-WM as 2-WM. Similar behaviors were observed for the 3-WM step response as the 2-WM. For all conditions an increase in blood pressure was observed in response to the sudden increase of blood flow rate. However the rate which pressure increased varied based on the condition with a highest rate for constricted conditions and the lowest rate for dilated conditions. However, the magnitude of the equilibrium value reached for each condition increased (Figure 8).

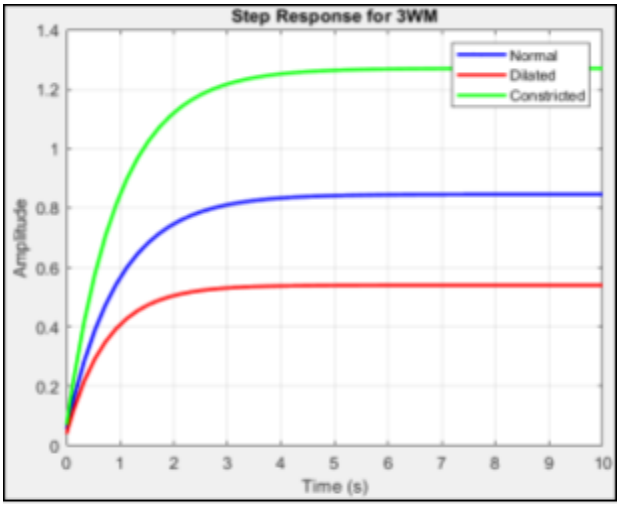

Figure 8: Normal, dilated, and constricted conditions for Step Response of 3 WM.

We then addressed the 2 and 3 WMs via a hemodynamic modeling approach to illustrate blood pressure dynamics. Understanding blood pressure dynamics involves a shift from the simplified 2-element model, where resistance and compliance are dominant factors, to the more complex 3-element model. The 3 WM introduces inertial effects, as demonstrated by the addition of the Z_C term, a crucial consideration for capturing the complexities of blood flow dynamics. In the 3-element model, inertial effects account for the acceleration and deceleration of blood flow by taking into account the mass of blood and the elasticity of arterial walls. The introduction of the characteristic resistance, represented as Z_C, refines the illustration of physiological phenomena such as wave reflections [1]. This factor becomes particularly significant when observing rapid changes in pressure for a given blood flow input. Illustrated in *Figure 9*, the normal, dilated, and constricted blood vessel conditions are impacted by the inertial effects during systole and diastole through the representation of the blood wave reflections. The 2-element model is represented by the solid red line, whereas the 3-element model is presented by the solid blue line. Though both models provide accurate representation of the blood pressure dynamics, the presence of Z_C refined the dynamics of blood pressure, capturing a more reflective quantitative analysis of pressure under the varying conditions. Additionally, based on the graphs, it is observed that the peak blood pressure–in mmHg– is around 106-109 for normal, 88 for dilated, and 142 for constricted blood vessel conditions. This aligns with the dynamic effects of vasodilator and vasoconstrictor drugs. As mentioned

before, vasodilators decrease resistance and increase compliance. Conversely, vasoconstrictors increase resistance and decrease compliance [7]. These effects are relevant to the graph when compared to normal conditions.

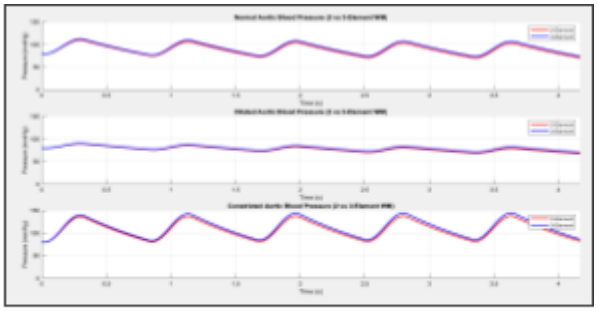

Figure 9: Normal, dilated, and constricted conditions for blood pressure output [mmHg].

For normal, dilated, and constricted conditions the aorta blood flow rate input frequency over time was modeled via MATLAB [5] for 5 cardiac cycles with a systole duration of 40% (Figure 10). For a typical heart rate of 75 beats per minute (BPM), for normal conditions, a stable blood flow can be observed with steady increase to blood flow peaks. For dilated conditions, bradycardia occurs where heart rate is slower, where 60 BPM [10]. Faster rise in blood flow during the systole was observed with sharper and higher peaks for blood flow rate. Lastly, tachycardia occurs when blood vessels are constricted with 100 BPM. Thus, leading to a slower rise in blood flow with lower peak flow rates compared to normal conditions.

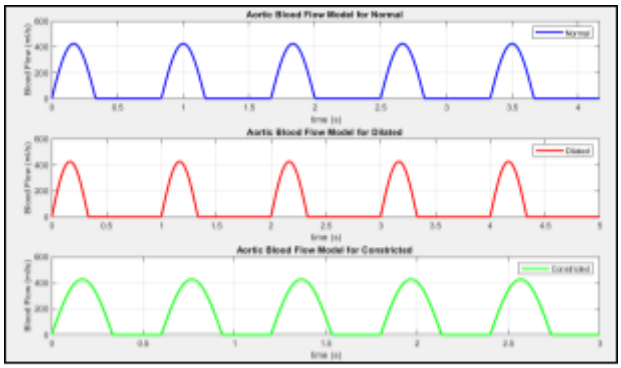

Figure 10: Simulated aortic blood flow I(t) for 5 cardiac cycles for normal, dilated, and constricted conditions.

From the derived Laplace transfer function (Eq. 2) and given parameters for each condition, the Bode plot was found for 2-WM. The frequency response exhibits a smooth magnitude roll-off, similar to a low-pass filter (Figure 11). Compared to the normal conditions, the gain (dB) increased for a constricted system due to the higher response amplitude. While for dilated conditions, the lower response amplitude decrease attributed to the gain decreased observed. The phase margin remains the constant for all conditions as they are modeled by the same system.

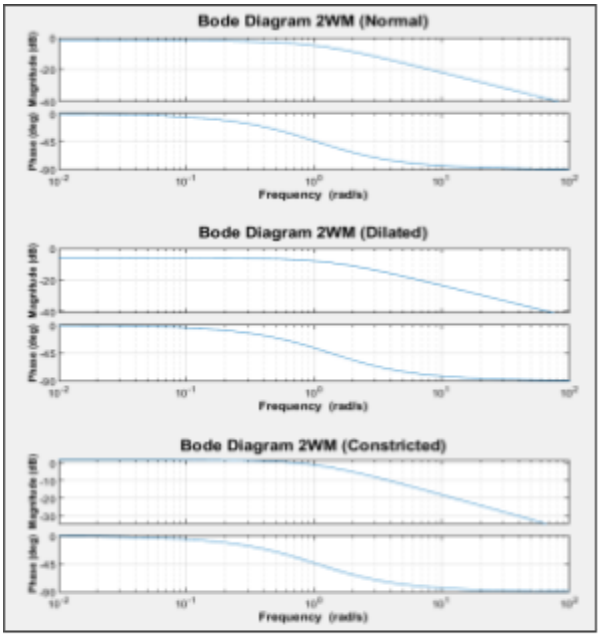

Figure 11: Bode diagram for normal, dilated, and constricted conditions using 2 WM.

The derived Lapace transfer function for a 3-element Windkessel model (Eq. 5) can be found in Figure 12). Similar to the frequency response observed in Figure 11, a shift up in gain occurred from higher response amplitude for constricted compared to normal conditions. Whereas, a shift down in gain was observed for dilated conditions in response to the lower response amplitude. The phase margin stayed constant due to the same modeled system applied. However, a phase shift reaching 90 degrees at resonant frequency and shift back to 0 degrees was observed. Similar to the 2-WM, the magnitude and phase response resemble a low-pass filter.

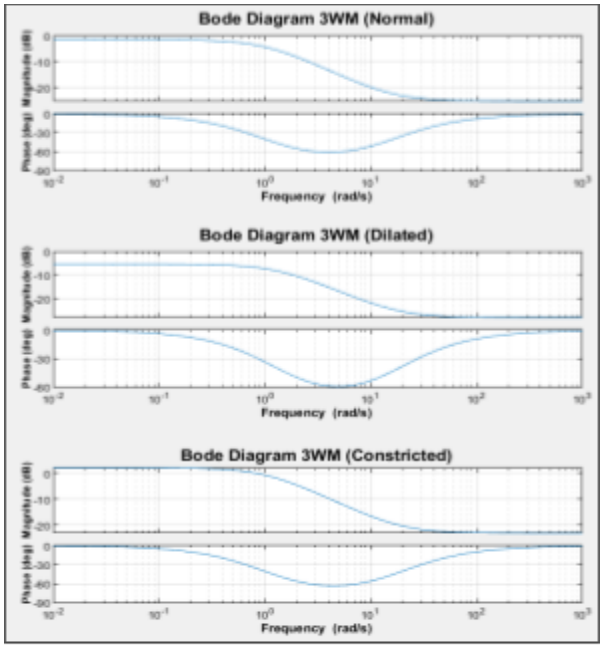

Figure 12: Bode diagram for normal, dilated, and constricted conditions using 3 WM.

III. DISCUSSION

The equation governing the 2-element Windkessel describes the dynamic of blood flow while the differential equations governing the 3-element Windkessel model show that 0 percent resistance to alternating blood flow is added. Utilizing the ordinary differential equations, we
can derive the Laplace transform function which is essential for analyzing blood flow signals across varying frequencies. These functions offer numerical information on input blood flow and output blood pressure response rate. They are allowing for a focused examination of blood pressure and cardiac output dynamics. This analysis in the frequency domain lays the groundwork for designing an effective control system to better understand and manage cardiovascular function.

The biggest difference between the 2 Simulink models is the inclusion of the characteristic resistance and the model would provide a more accurate simulation of the cardiac cycle.

We compared the step responses of the cardiovascular system model. The Simulink plot of the 2-element Windkessel model revealed a lower step response than the 3-element Windkessel model, highlighting the latter's enhanced ability to capture the dynamic changes to the system. This can be attributed to how the 2-element model lacks the atrial inertia compared to the 3-element model, which then results in a more realistic response.

We analyzed the output response for the blood pressure of the 3 blood vessel conditions: normal, dilated, and constricted. For the 2 element model, blood pressure dynamics were primarily influenced by resistance and compliance. The 3 element model introduced the inertial flux accounting for the acceleration and deceleration of blood flow. This component, cc, considers the mass of blood and the elasticity of the arterial wall, deriving a more comprehensive representation of blood pressure output. Incorporating the characteristic resistance enhances the ability to simulate more physiological phenomenons such as wave functions and more accurately captures the dynamics of blood flow, especially when rapid changes occur. As seen in the normal, dilated, and constricted figures, we can see the impact of the inertial effects on diastole and systole via the wave function. The effect of the vasodilator drugs showed a decrease in pressure while the vasoconstrictor drug showed a more oscillated and increased blood pressure.

The input pulse frequency was compared per condition. The aortic blood flow was simulated where a heart rate of 75 beats per minute (BPM) was used for a normal cardiovascular system. In dilated conditions, bradycardia, a slow heart rate, can occur, resulting in a heart rate of 60 BPM. In constricted conditions, tachycardia, a fast heart rate, can transpire, resulting in a heart rate of 100 BPM. The normal condition denotes the baseline comparative condition. For the dilated system, a lower frequency response was evaluated, resulting in a decreased magnitude depicted in the shift of 0 dB to -5 dB as seen in Figures 11 and 12. . This suggests that for a cardiovascular system where the blood vessels are dilated, a slower response and decrease in blood pressure is observed. However, for constricted conditions, a higher frequency response was considered. This was represented by the increase in magnitude in Figures # and # (Figure # for bode plots), where a shift up from 0 dB to 1 dB was observed, indicating a faster response hence an increase in blood pressure.

Lastly, from the derived Laplace transform functions, and the given parameters for each condition, the bode plot can be found to depict the frequency response for both the 2 and 3-element Windkessel models. The 2-element model exhibits a smoother magnitude role, while the 3-element model introduces complexity with resonance effects as indicated by phase

shift reaching 90 degrees at the resonance frequency and back to 0 degrees. The magnitude change can be represented by the physiological changes of the blood vessel according to the parameters, like the dilated and constricted vessels. It is also key to mention that the bode plots resemble the behavior of a low pass filter, which indicates a stable system for our model.

IV. CONCLUSION

The Windkessel model offers a thorough comprehension of the circulatory system's dynamics. Through the application of the 2-WM and 3-WM model, the differential equations were and the Laplace transforms were derived to establish transfer functions that model the blood flow and pressure in the cardiovascular system. MATLAB and Simulink were used in parallel to model and analyze the Windkessel Model. The normal, dilated, and constricted blood flow and pressure conditions are simulated using the two and three-element Windkessel models. The models use peripheral resistance, inertia, and arterial compliance to calculate blood pressure and blood flow correlations. The comparison between the two models with the various conditions revealed that the 3-WM offered a more accurate representation of the cardiovascular system, by incorporating the characteristic resistances. Important information about the effects of vasoactive medications on the circulatory system can be gained from the computational study of blood flow and pressure response rates. A valuable tool for comprehending and controlling cardiovascular function is the Windkessel model. The results from the study offer a foundation for theoretical and practical application in cardiovascular drug research.

V. CLINICAL APPLICATIONS

Blood dynamic modeling, particularly using the refined 2 and 3 WM, takes into account the effects of resistance, compliance, and inertia. These characteristics hold significant promise for clinical applications in cardiovascular health. This modeling approach enables the monitoring of cardiovascular pathologies, offering understanding of blood pressure dynamics and facilitating timely intervention. Moreover, the simulation of pharmacological interventions, such as vasodilators and vasoconstrictors, provides valuable insights for tailoring treatment strategies. In clinical applications, the use of the Windkessel model can serve to personalize treatment plans for various hemodynamic profiles, optimizing cardiovascular treatment and improving patient outcomes.

VI. ACKNOWLEDGEMENT

We thank Professor Gert Cauwenberghs and teaching assistants Benjamin Balster and Tatiana Kassem for their instruction and support throughout the quarter.

VII. REFERENCES

[1] R, K. Arterial dilation and reduced wave reflection. Benefit of dilevalol in hypertension. AHA Journals. (1989). https://doi.org/10.1161/01.HYP.14.1.14

[2] Safar, M. E.. Windkessel Effect. Science Direct (2007) https://doi.org/10.1016/B978-0-323-03961-1.50016-7

[3] Sica, D. A. (2007). Direct Vasodilators and their Role in Hypertension Management: Minoxidil. National Library of Medicine. https://doi.org/10.1111%2Fj.1524-6175.2001.00455.x

[4] Westerhof, N et al. “An artificial arterial system for pumping hearts.” Journal of applied physiology vol. 31,5 (1971): 776-81. doi:10.1152/jappl.1971.31.5.776

[5] Hlavac, M et al. “Windkessel Model Analysis in MATLAB”. https://doi.org/https://www.semanticscholar.org/paper/WINDKESSEL-MODEL-ANALYSIS-IN-MATLAB-Hlav%C3%A1%C4%8D/e8fa00bad26b7a549a1834bf4d7a1422882ff514

[6] Ruel, J., & Lachance, G. (2010). Mathematical modeling and experimental testing of three bioreactor configurations based on windkessel models. Heart international, 5(1), e1. https://doi.org/10.4081/hi.2010.e1

[7] Deckerman, J. et al. “Difference Between Vasoconstriction and Vasodilating Medications' '. *WedMD*. https://doi.org/https://www.webmd.com/hypertension-high-blood-pressure/difference-between-vasoconstricting-vasodilating-medications

[8] Juila, K. (2023). *Vasoconstriction and vasodilation* [Photograph].

[9] Windkessel effect. (2023, November 8). In *Wikipedia*. https://en.wikipedia.org/wiki/Windkessel_effect

[10] Autonomic and endocrine control of cardiovascular function. *NIH*. https://doi.org/10.4330%2Fwjc.v7.i4