BENG 122A Fall 2020

Quiz 1

Tuesday, November 3, 2020

Name *(Last, First):* SOLUTIONS

- This quiz is open book, open notes, and online, but web search is prohibited. You may follow electronic links from Canvas or the class web pages, but not any further. **No collaboration or communication in any form is allowed**, except for questions to the instructor and TAs.
- The quiz is due November 4, 2020 at 11:59pm, over Canvas. It should approximately take 2 hours to complete, but there is no time limit other than the submission deadline. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.
- There are 3 problems. Points for each problem are given in **[brackets]**. There are 100 points total.

1. **[35 pts]** Consider the following biochemical reaction taking place in an organ in the body:

$$\mathrm{A} + \mathrm{B} \underset{k_r}{\overset{k_f}{\rightleftharpoons}} \mathrm{C} + \mathrm{D}$$

where compounds A and B combine to generate compounds C and D at rate k_f, and C and D recombine to regenerate A and B at rate k_r.

(a) [5 pts] Show that if compounds B and C are present at much higher concentrations than either A or D, then you may assume that their concentrations are approximately constant, $[\mathrm{B}] \approx [\mathrm{B}]_0$ and $[\mathrm{C}] \approx [\mathrm{C}]_0$.

$$\frac{d}{dt}[A] = \frac{d}{dt}[B] = -\frac{d}{dt}[C] = -\frac{d}{dt}[D] =$$

$$- k_f [A][B] + k_r [C][D]$$

Every mol change in B or C requires a mol change of same magnitude in A and in D, where A and D move in opposite directions.

Therefore, and because A and D are always positive, B and C cannot change more than the larger amount of A and D currently present.

If B and C are present in much greater amounts, then the relative change in B and C must be very small.

Hence we can approximate [B] and [C] to be relatively constant.

$$[B]_0 - \underbrace{[A]_0}_{\ll [B]_0} \leq [B] \leq [B]_0 + \underbrace{[D]_0}_{\ll [B]_0} \Rightarrow [B] \approx [B]_0$$

$$[C]_0 - \underbrace{[D]_0}_{\ll [C]_0} \leq [C] \leq [C]_0 + \underbrace{[A]_0}_{\ll [C]_0} \Rightarrow [C] \approx [C]_0$$

(b) [10 pts] Under those assumptions, and further assuming that both A and D exit the volume V of the organ at a flow rate Q, while B and C recirculate in the organ without decay, write the ODEs in the concentrations [A] and [D] that describe both the kinetics and the flow.

$$\begin{cases} \frac{d}{dt}[A] = -\frac{Q}{V}[A] - k_f [A][B]_0 + k_r [C]_0[D] \\ \frac{d}{dt}[D] = -\frac{Q}{V}[D] + k_f [A][B]_0 - k_r [C]_0[D] \end{cases}$$

Let: $a = \frac{Q}{V}$

$b = k_f [B]_0$

$c = k_r [C]_0$ constants

$$\begin{cases} \frac{d}{dt}[A] = -(a+b)[A] + c[D] \\ \frac{d}{dt}[D] = b[A] - (a+c)[D] \end{cases}$$

(c) [5 pts] Find the equilibrium (*i.e.*, the steady-state) concentrations.

$$\begin{cases} \frac{d}{dt}[A] = -(a+b)[A] + c[D] = 0 \\ \frac{d}{dt}[D] = b[A] - (a+c)[D] = 0 \end{cases}$$

$$a \neq 0 \Rightarrow [A]_{s.s.} = [D]_{s.s.} = 0$$

and $[B]_{s.s} = [B]_0$

$[C]_{s.s.} = [C]_0$

(d) [15 pts] Use Laplace transforms to find the concentration [A] as a function of time, starting from initial conditions $[\mathrm{A}](0) = [\mathrm{A}]_0$ and $[\mathrm{D}](0) = 0$.

$$\begin{cases} s\,[A](s) - [A]_0 = -(a+b)\,[A](s) + c\,[D](s) \\ s\,[D](s) - \underbrace{[D]_0}_{=0} = b\,[A](s) - (a+c)\,[D](s) \end{cases}$$

$$\begin{cases} (s+a+b)\,[A](s) = [A]_0 + c\,[D](s) \\ (s+a+c)\,[D](s) = b\,[A](s) \end{cases}$$

$$(s+a+b)(s+a+c)\,[A](s) = (s+a+c)\,[A]_0 + bc\,[A](s)$$

$$[A](s) = \frac{s+a+c}{(s+a+b)(s+a+c) - bc}\,[A]_0 = \frac{s+a+c}{(s+a)(s+a+b+c)}\,[A]_0$$

$$[A](s) = \frac{1}{b+c}\left(\frac{c}{s+a} + \frac{b}{s+a+b+c}\right)[A]_0$$

$$[A](t) = \frac{1}{b+c}\left(c\,e^{-at} + b\,e^{-(a+b+c)t}\right)[A]_0$$

$$[A](t) = \frac{k_2[C]_0 + k_f[B]_0\,e^{-(k_2[C]_0 + k_f[B]_0)t}}{k_2[C]_0 + k_f[B]_0}\,e^{-\frac{Q}{V}t}\,[A]_0$$

2. **[40 pts]** Consider the following set of ODEs describing the dynamics of a biomechanical system with mass m and damping γ, with force $f(t)$ driving the input, and with position $u(t)$ at the output:

$$
\begin{aligned}
\frac{du}{dt} &= v(t) \\
m\,\frac{dv}{dt} &= -\gamma\, v(t) + f(t)\,.
\end{aligned}
$$

(a) [5 pts] Find the Laplace transform of position $u(s)$ as a function of the Laplace transform of the force $f(s)$, and initial conditions in position $u(0) = u_0$ and velocity $v(0) = v_0$.

$$s\,u(s) - u_0 = v(s)$$

$$m\left(s\,v(s) - v_0\right) = -\gamma\, v(s) + f(s)$$

$$m\left(s\left(s\,u(s) - u_0\right) - v_0\right) = -\gamma\left(s\,u(s) - u_0\right) + f(s)$$

$$\left(m s^2 + \gamma s\right) u(s) = m u_0 s + m v_0 + \gamma u_0 + f(s)$$

$$u(s) = \underbrace{\frac{m u_0 s + m v_0 + \gamma u_0}{s(ms + \gamma)}}_{\substack{\text{I.C.} \\ 2(b)}} + \underbrace{\frac{1}{s(ms + \gamma)}}_{\substack{H(s) \\ 2(c)}}\, f(s)$$

(b) [10 pts] For zero force $f(t) \equiv 0$, and for given initial conditions $u(0) = u_0$ and $v(0) = v_0$, find the position $u(t)$ as a function of time.

$$u(s) = \frac{m u_0 \cancel{s}}{(ms+\gamma)\cancel{s}} + \frac{m v_0 + \gamma u_0}{(ms+\gamma)s} \qquad (f(s) = 0)$$

$$= \frac{u_0}{s + \frac{\gamma}{m}} + \left(v_0 + \frac{\gamma}{m} u_0\right) \underbrace{\frac{1}{\left(s + \frac{\gamma}{m}\right)s}}_{= \frac{m}{\gamma}\left(\frac{1}{s} - \frac{1}{s+\frac{\gamma}{m}}\right)}$$

$$= \frac{u_0}{s+\frac{\gamma}{m}} + \left(\frac{m}{\gamma} v_0 + u_0\right)\left(\frac{1}{s} - \frac{1}{s+\frac{\gamma}{m}}\right)$$

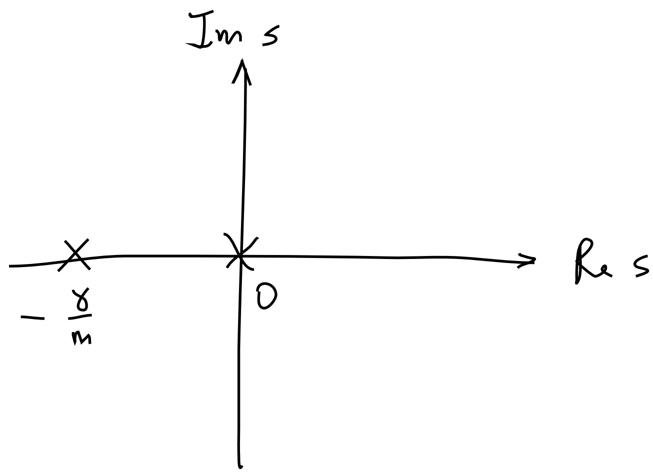
$$= \frac{u_0}{s} + \frac{m}{\gamma} v_0 \left(\frac{1}{s} - \frac{1}{s+\frac{\gamma}{m}}\right)$$

$$u(t) = u_0 + \frac{m}{\gamma} v_0 \left(1 - e^{-\frac{\gamma}{m}t}\right)$$

(c) [5 pts] Find the transfer function $H(s) = u(s)/f(s)$ of the system, and find the poles and zeros.

$$H(s) = \frac{u(s)}{f(s)} = \frac{1}{s(ms+\gamma)}$$

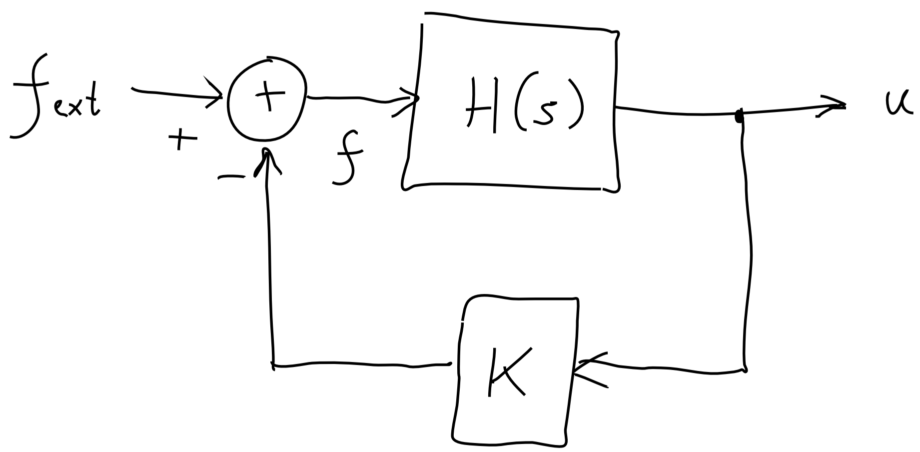
- ZEROS: NONE
- POLES: $\begin{cases} p_1 = 0 \\ p_2 = -\frac{\gamma}{m} \end{cases}$



(d) [10 pts] Now consider closed-loop feedback, in which the force $f(t)$ is given by

$$f(t) = f_{ext}(t) - K\,u(t)$$

where $f_{ext}(t)$ is the externally applied force, and K is the feedback gain. Draw the closed-loop system block diagram, and find the closed-loop transfer function $F(s) = u(s)/f_{ext}(s)$.



$$u(s) = H(s)\,f(s) = H(s)\left(f_{ext}(s) - K\,u(s)\right)$$

$$F(s) = \frac{u(s)}{f_{ext}(s)} = \frac{H(s)}{1 + K H(s)}$$

$$= \frac{1}{ms^2 + \gamma s + K}$$

(e) [10 pts] Find the value of the feedback gain K that minimizes the settling time of the closed-loop system. *Hint*: you may make approximate arguments. Think of settling time in terms of the poles of the system.

Poles: $m s^2 + \gamma s + K = 0$

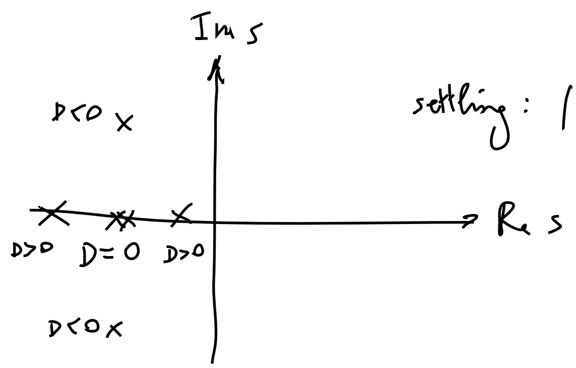
$$p_{\frac{1}{2}} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4Km}}{2m}$$

Overdamped (discriminant D > 0): Two real poles; slow settling and rise time.

Critically dampled (D = 0): Two identical poles; fast settling and rise time.

Underdamped (D < 0): Two complex conjugate poles; faster rise time, but not faster settling.

The critically damped case gives the fastest settling, without ringing.



settling: $|x(t) - x_\infty| \sim e^{\mathrm{Re}\, p_i \cdot t}$

$$D = \gamma^2 - 4Km = 0 \quad \Rightarrow \quad K = \frac{\gamma^2}{4m}$$

3. **[25 pts]** Linear time invariant biosystems:

(a) [5 pts] Give two possible explanations of biophysical phenomena that cause non-zero pressure across a vessel.

1. Resistance in flow due to blood viscosity
2. Height differences under gravity

Others (many are possible!):

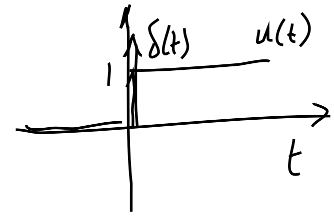
3. Velocity nonuniformity due to differences in diameter
4. Active pumping (forces acting on the vessel walls)

etc...

(b) [10 pts] What is the time integral of the Delta-Dirac impulse? And what is the time integral of the impulse response of a linear time-invariant system? Explain the connection between the two.

The time integral of the Delta-Dirac impulse is a step function.

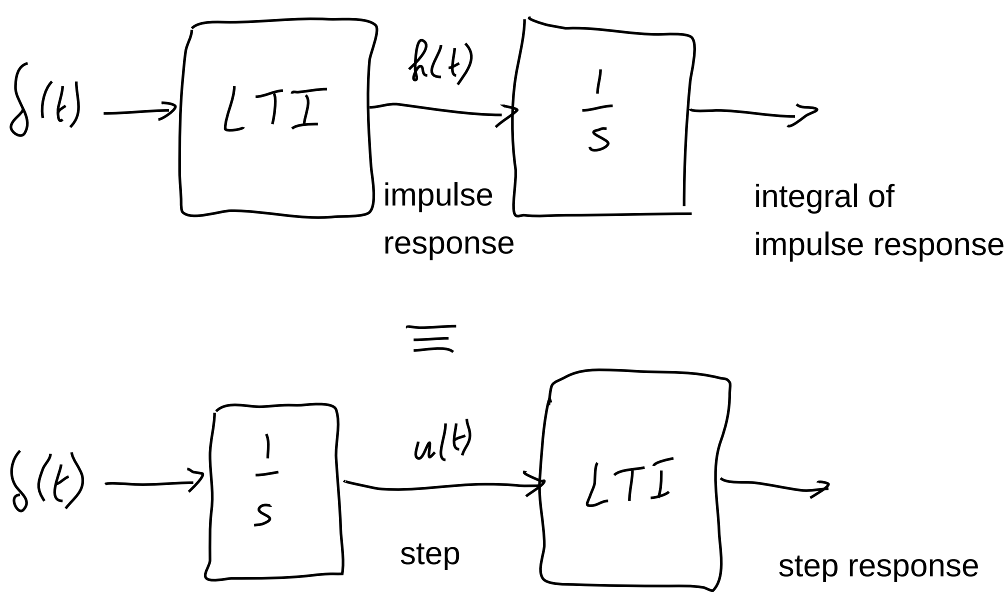
$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0 \end{cases}$$



Alternatively:

$$\underbrace{1}_{\mathcal{L}(\delta(t))} \longrightarrow \underbrace{\boxed{\frac{1}{s}}}_{\text{INTEGRATOR}} \longrightarrow \underbrace{\frac{1}{s}}_{\mathcal{L}(u(t))}$$

The time integral of the impulse response of a LTI system is the step response of that LTI system.



(c) [10 pts] Show that the linear time-invariant system given by the ODE

$$\frac{du}{dt} = A\,u(t) + f(t) \tag{1}$$

with initial condition $u(0) = u_0$ is equivalent, for all strictly positive times $t > 0$, to a system with the same ODE and zero initial conditions $u(0) = 0$ but with modified force $f(t) + u_0\,\delta(t)$, *i.e.,* by absorbing the initial condition as an initial "kick" in the force. *Hint*: consider left and right limits of $u(t)$ at time $t = 0$.

The effect of the "kick" is to step the state variable, from zero to the same level as the initial condition:

$$\underbrace{u(0^+)}_{\text{immediately after } t=0} = \underbrace{u(0^-)}_{\text{immediately before } t=0} + \underbrace{\int_{0^-}^{0^+} \left(A\,u(t) + f(t) + u_0\,\delta(t)\right) dt}_{= u_0\,!}$$

Alternatively:

$$s\,u(s) - 0 = A\,u(s) + \Big(f(s) + \underbrace{u_0 \cdot 1}_{\mathcal{L}(\delta(t))}\Big)$$

is equivalent to:

$$s\,u(s) - \underbrace{u_0}_{\substack{\text{I.C.}\\ u(0)}} = A\,u(s) + f(s)$$