BENG 122A Fall 2021

Quiz 1

Tuesday, October 26, 2021

Name *(Last, First):* SOLUTIONS

- This quiz is open book, open notes, and online, but web search is prohibited. You may follow electronic links from Canvas or the class web pages, but not any further. **No collaboration or communication in any form is allowed**, except for questions to the instructor and TAs.
- The quiz is due October 27, 2021 at 11:59pm, over Canvas. It should approximately take 2 hours to complete, but there is no time limit other than the submission deadline. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.
- There are 3 problems. Points for each problem are given in **[brackets]**. There are 100 points total.

1. **[35 pts]** Consider the following biochemical reaction taking place in an organ in the body:

$$\mathrm{A} + \mathrm{B} \underset{k_r}{\overset{k_f}{\rightleftharpoons}} \mathrm{C}$$

where compounds A and B combine to generate compound C at rate k_f, and C decomposes to regenerate A and B at rate k_r. Compounds B and C are present at much higher concentrations than A, so we assume that their concentrations remain approximately constant, $[\mathrm{B}] \approx [\mathrm{B}]_0$ and $[\mathrm{C}] \approx [\mathrm{C}]_0$. Compound A exits the volume V of the organ at a flow rate Q, while B and C recirculate in the organ without decay.

   (a) [10 pts] Under these approximating assumptions, write the ODE in the concentration $[\mathrm{A}]$ that describes both the reaction kinetics and the flow.

$$\frac{d[A]}{dt} = \underbrace{-\frac{Q}{V}\cdot[A]}_{\text{outflow}} \underbrace{- k_f [A]\cdot[B]_0 + k_r [C]_0}_{\text{reaction kinetics}}$$

(b) [5 pts] Find the equilibrium (*i.e.*, the steady-state) concentrations.

At steady state: $\frac{d[A]}{dt} = 0$ and thus:

$$-\frac{Q}{V}[A]_{ss} - k_f [A]_{ss}[B]_0 + k_r [C]_0 = 0$$

$$[A]_{ss} = \frac{k_r [C]_0}{\frac{Q}{V} + k_f [B]_0}$$

Also:

$$[B]_{ss} = [B]_0$$

$$[C]_{ss} = [C]_0$$

(c) [15 pts] Use Laplace transforms to find the concentration $[\mathrm{A}]$ as a function of time, starting from initial conditions $[\mathrm{A}](0) = [\mathrm{A}]_0$.

$$s[A](s) - [A]_0 = -\left(\frac{Q}{V} + k_f[B]_0\right)[A](s) + \underbrace{k_r[C]_0 \cdot \frac{1}{s}}$$

A constant activation
amounts to a step
in Lapace

$$[A](s) = \frac{[A]_0}{s + \left(\frac{Q}{V} + k_f[B]_0\right)} + \frac{k_r[C]_0}{s\left(s + \left(\frac{Q}{V} + k_f[B]_0\right)\right)}$$

Inverse Laplace (from Laplace tables):

$$[A](t) = [A]_0\, e^{-\left(\frac{Q}{V} + k_f[B]_0\right)t} + \frac{k_r[C]_0}{\frac{Q}{V} + k_f[B]_0}\left(1 - e^{-\left(\frac{Q}{V} + k_f[B]_0\right)t}\right)$$

(d) [5 pts] Is the above approximating assumption that $[B] \approx [B]_0$ and $[C] \approx [C]_0$ reasonable under these conditions? Explain (in words) to what extent you expect your answers in (b) and (c) to change accounting for the reaction kinetics in compounds B and C.

Not quite. Since the supply of C is finite but A is continually depleted by flowing outside of the organ, all of C will eventually be converted to A and B, with the added B retained in the organ, but all of A decaying by exiting the organ.

Specifically (beyond words):

$$\frac{d[B]}{dt} = -\frac{d[C]}{dt} = -k_f[A][B] + k_r[C]$$

$$\frac{d[A]}{dt} = -\frac{Q}{V}[A] + \frac{d[B]}{dt}$$

At steady state: $\frac{d}{dt} \approx 0$

$$\Rightarrow \quad [A]_{ss} = 0$$

$$[B]_{ss} = [B]_0 + [C]_0$$

$$[C]_{ss} = 0$$

2. **[45 pts]** Consider the following set of ODEs describing the dynamics of a biomechanical system with mass m, stiffness k, and damping γ, with force $f(t)$ driving the input, and with position $u(t)$ at the output:

$$\begin{aligned} \frac{du}{dt} &= v(t) \\ m\,\frac{dv}{dt} &= -\gamma\, v(t) - k\, u(t) + f(t)\,. \end{aligned}$$

(a) [5 pts] Find the Laplace transform of position $u(s)$ as a function of the Laplace transform of the force $f(s)$, for zero initial conditions in position $u(0) = 0$ and velocity $v(0) = 0$.

$$s\,u(s) - \cancel{u(0)} = v(s)$$

$$m\left(s\,v(s) - \cancel{v(0)}\right) = -\gamma\, v(s) - k\,u(s) + f(s)$$

$$\Rightarrow \quad m\left(s\,(s\,u(s))\right) = -\gamma\left(s\,u(s)\right) - k\,u(s) + f(s)$$

$$\left(m s^2 + \gamma s + k\right) u(s) = f(s)$$

$$u(s) = \underbrace{\frac{1}{m s^2 + \gamma s + k}}_{H(s) \text{ in } 2(c)} \cdot f(s)$$

(b) [15 pts] Find the step response of the system in two extreme cases: *i*) zero damping $\gamma = 0$; and *ii*) zero stiffness $k = 0$. The step response is the position $u(t)$ for a unit step in the force, $f(t) = 1$ for $t > 0$, from zero initial conditions. Explain the differences you observe in the two cases.

$$u(s) = \frac{1}{ms^2+\gamma s+k} f(s) = \frac{1}{ms^2+\gamma s+k} \cdot \frac{1}{s}$$

$$i) \quad u(s) = \frac{1}{s(ms^2+k)} = \frac{1}{m} \cdot \frac{1}{s} \cdot \frac{1}{s^2+\frac{k}{m}}$$

Solution by partial fraction decomposition:

$$u(s) = \frac{1}{k}\frac{1}{s} - \frac{1}{k}\frac{s}{s^2+\frac{k}{m}} \xRightarrow[\text{tables}]{\text{Laplace}} u(t) = \frac{1}{k}\left(1 - \cos\left(\sqrt{\frac{k}{m}}t\right)\right)$$

Solution by integration:

$$u(s) = \frac{1}{s} \cdot \left(\frac{1}{m} \cdot \frac{1}{s^2+\frac{k}{m}}\right) \Rightarrow u(t) = \int_0^t \frac{1}{m} \cdot \left(\sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}t\right)\right) dt$$

$$= \frac{1}{\cancel{m}} \cdot \frac{\cancel{m}}{k}\left[-\cos\left(\sqrt{\frac{k}{m}}t\right)\right]_0^t \quad \text{same!}$$

$$ii) \quad u(s) = \frac{1}{s^2(ms+\gamma)} = \frac{1}{m} \cdot \frac{1}{s^2} \cdot \frac{1}{s+\frac{\gamma}{m}}$$

Solution by partial fraction decomposition:

$$u(s) = \frac{1}{\gamma}\frac{1}{s^2} - \frac{m}{\gamma^2}\frac{1}{s} + \frac{m}{\gamma^2}\frac{1}{s+\frac{\gamma}{m}} \Rightarrow u(t) = \frac{1}{\gamma}t - \frac{m}{\gamma^2} + \frac{m}{\gamma^2}e^{-\frac{\gamma}{m}t}$$

Solution by integration:

$$u(s) = \frac{1}{s^2} \cdot \left(\frac{1}{m} \cdot \frac{1}{s+\frac{\gamma}{m}}\right) \xRightarrow[\text{twice}]{\text{integrate}} u(t) = \int_0^t\int_0^\vartheta \left(\frac{1}{m} e^{-\frac{\gamma}{m}\vartheta}\right) d\vartheta \, dt$$

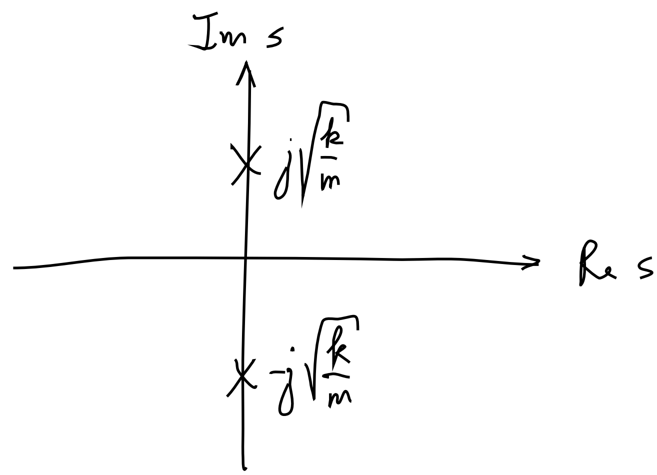
$$= \int_0^t \frac{1}{m} \cdot \frac{m}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right) dt = \frac{1}{\gamma}t - \frac{m}{\gamma^2}\left(1 - e^{-\frac{\gamma}{m}t}\right) \quad \text{same!}$$

i) undergoes sustained oscillations whereas ii) settles into a linear ramp at constant drag velocity.

(c) [10 pts] Find the transfer function $H(s) = u(s)/f(s)$ of the system, and find the poles and zeros in the two extreme cases *i*) zero damping $\gamma = 0$ and *ii*) zero stiffness $k = 0$. Compare the stability of the system in the two cases.

$$H(s) = \frac{u(s)}{f(s)} = \frac{1}{ms^2 + \gamma s + k}$$

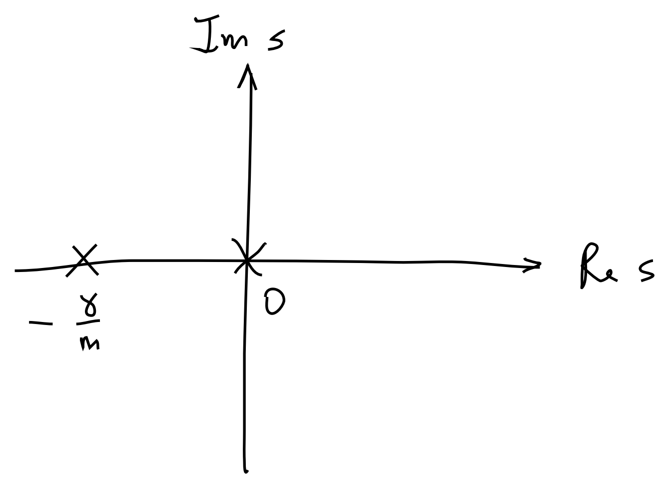
$$i)\quad H(s) = \frac{1}{ms^2 + k}$$



- no zeros

- poles: $\begin{cases} p_1 = +j\sqrt{\frac{k}{m}} \\ p_2 = -j\sqrt{\frac{k}{m}} \end{cases}$

$$ii)\quad H(s) = \frac{1}{s(ms + \gamma)}$$



- no zeros

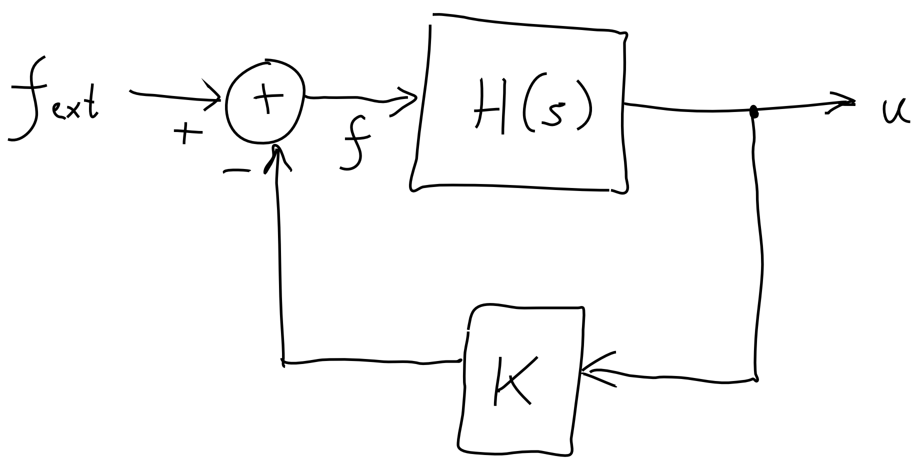
- poles: $\begin{cases} p_1 = 0 \\ p_2 = -\frac{\gamma}{m} \end{cases}$

Both are critically stable, with poles on the imaginary axis, and no poles to the right of the imaginary axis.

(d) [10 pts] Now consider closed-loop feedback, in which the force $f(t)$ is given by

$$f(t) = f_{ext}(t) - K\,u(t)$$

where $f_{ext}(t)$ is the externally applied force, and K is the feedback gain. Draw the closed-loop system block diagram, and find the closed-loop transfer function $F(s) = u(s)/f_{ext}(s)$. Show that the feedback gain K in the closed-loop system is equivalent to extra stiffness k in the open-loop system.



$$u(s) = H(s)f(s) = H(s)\left(f_{ext}(s) - K\,u(s)\right)$$

$$F(s) = \frac{u(s)}{f_{ext}(s)} = \frac{H(s)}{1 + K H(s)}$$

$$= \frac{1}{ms^2 + \gamma s + \underbrace{(k+K)}_{\text{Effective stiffness}}}$$

(e) [5 pts] Find the value of the feedback gain K for which the closed-loop system is critically damped. What does this mean in terms of the settling time?

Poles: $$ms^2 + \gamma s + k + K = 0$$

$$p_{1,2} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4(k+K)m}}{2m}$$

Critical damping: $$\gamma^2 - 4(k+K)m = 0$$

$$\Rightarrow \quad K = \frac{\gamma^2}{4m} - k$$

The critically damped setting for K gives the fastest settling, without ringing. (Increasing K beyond that gives faster rise time, but longer settling time due to increased ringing.)

3. **[20 pts]** Linear time invariant and conservative biosystems:

(a) [5 pts] In a series connection of multiple blocks of linear time invariant systems, does it matter in which order the blocks are connected? Explain.

No, because the transfer function is the product of the transfer functions, and the product is commutative (the order of multiplication doesn't matter).

(b) [5 pts] Show that the step response of a linear time invariant system is the time integral of its impulse response. *Hint*: consider the setting of Problem 3 (a) where one of the blocks is an integrator with transfer function $\frac{1}{s}$.

Step response:

$$U(s) = H(s) \cdot \underbrace{\frac{1}{s}}_{\text{Laplace of step}} = \underbrace{\frac{1}{s}}_{\text{Integrator}} \cdot H(s) \cdot \underbrace{1}_{\text{Laplace of impulse}}$$

Hence in the time domain:

$$\underbrace{u(t)}_{\text{step response}} = \int_0^t \underbrace{h(t)}_{\text{impulse response}} dt$$

(c) [5 pts] To what extent does pressure in a blood vessel depend on altitude? Explain.

Potential energy, and thus pressure, increases with height as:

$$\Delta p = \rho \cdot g \cdot \Delta h$$

(d) [5 pts] To what extent does pressure increase or decrease with increasing diameter of the blood vessel? Explain.

Kinetic energy, and thus pressure, increases with velocity as:

$$\Delta p = \frac{\rho}{2} \cdot \Delta(v^2)$$

At constant flow rate, velocity decreases with diameter, and hence pressure decreases with increasing diameter.