BENG 122A Fall 2024

Quiz 1

Tuesday, October 29, 2024

Name *(Last, First):* SOLUTIONS

- This quiz is open book, open notes, and online, but web search is prohibited. You may follow electronic links from Canvas or the class web pages, but not any further. **No collaboration or communication in any form is allowed**, except for questions to the instructor and TAs.
- The quiz is due October 31, 2024 at 11:59pm, over Canvas. It should approximately take 2 hours to complete, but there is no time limit other than the submission deadline. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.
- There are 3 problems. Points for each problem are given in **[brackets]**. There are 100 points total.

1. **[40 pts]** Consider the following biochemical reaction taking place in an organ in the body:

$$\mathrm{A} + \mathrm{B} \underset{k_r}{\overset{k_f}{\rightleftharpoons}} \mathrm{C}$$

where compounds A and B combine to generate compound C at rate k_f, and C decomposes to regenerate A and B at rate k_r. Compound B is present at much higher concentration than A and C, so we assume that its concentration remains approximately constant, $[\mathrm{B}] \approx [\mathrm{B}]_0$. Compounds A and C exit the volume V of the organ at a flow rate Q, while B recirculates in the organ without decay.

   (a) [10 pts] Under these approximating assumptions, write the ODEs in the concentrations [A] and [C] that describe both the reaction kinetics and the flow.

$$\frac{d[A]}{dt} = -\frac{Q}{V}[A] - k_f [A][B]_0 + k_r [C]$$

$$\frac{d[C]}{dt} = -\frac{Q}{V}[C] + k_f [A][B]_0 - k_r [C]$$

(b) [5 pts] Find the equilibrium (*i.e.*, the steady-state) concentrations.

$$0 = -\frac{Q}{V}\overline{[A]} - k_f \overline{[A]}\,[B]_0 + k_r \overline{[C]}$$

$$0 = -\frac{Q}{V}\overline{[C]} + k_f \overline{[A]}\,[B]_0 - k_r \overline{[C]}$$

$$\left(\frac{Q}{V} + k_f [B]_0\right)\overline{[A]} = k_r \overline{[C]}$$

$$k_f [B]_0 \overline{[A]} = \left(\frac{Q}{V} + k_r\right)\overline{[C]}$$

$\frac{Q}{V} \neq 0$

$$\Rightarrow \quad \overline{[A]} = \overline{[C]} = 0$$

(c) [25 pts] Use Laplace transforms to find the concentrations [A] and [C] as a function of time, starting from initial conditions $[\mathrm{A}](0) = [\mathrm{A}]_0$ and $[\mathrm{C}](0) = [\mathrm{C}]_0$.

$$s[A] - [A]_0 = -\frac{Q}{V}[A] - k_f[B]_0[A] + k_r[C]$$

$$s[C] - [C]_0 = -\frac{Q}{V}[C] + k_f[B]_0[A] - k_r[C]$$

$$\begin{pmatrix} s + \frac{Q}{V} + k_f[B]_0 & -k_r \\ -k_f[B]_0 & s + \frac{Q}{V} + k_r \end{pmatrix} \cdot \begin{pmatrix} [A] \\ [C] \end{pmatrix} = \begin{pmatrix} [A]_0 \\ [C]_0 \end{pmatrix}$$

$$\det \cdot \begin{pmatrix} [A] \\ [C] \end{pmatrix} = \begin{pmatrix} s + \frac{Q}{V} + k_r & k_r \\ k_f[B]_0 & s + \frac{Q}{V} + k_f[B]_0 \end{pmatrix} \cdot \begin{pmatrix} [A]_0 \\ [C]_0 \end{pmatrix}$$

with determinant:

$$\begin{aligned} \det &= \left(s + \frac{Q}{V} + k_r\right)\left(s + \frac{Q}{V} + k_f[B]_0\right) - k_r k_f [B]_0 \\ &= \left(s + \frac{Q}{V}\right)^2 + \left(k_f[B]_0 + k_r\right)\left(s + \frac{Q}{V}\right) + \cancel{k_r k_f [B]_0} - \cancel{k_r k_f [B]_0} \\ &= \left(s + \frac{Q}{V}\right)\left(s + \frac{Q}{V} + k_f[B]_0 + k_r\right) \end{aligned}$$

$$[A](s) = \frac{\left(s+\frac{Q}{V}\right)[A]_0 + k_r\left([A]_0+[C]_0\right)}{\left(s+\frac{Q}{V}\right)\left(s+\frac{Q}{V}+k_f[B]_0+k_r\right)}$$

Partial fraction decomposition:

$$= \frac{1}{k_f[B]_0+k_r}\left(\frac{k_r\left([A]_0+[C]_0\right)}{s+\frac{Q}{V}} + \frac{k_f[B]_0[A]_0+k_r[C]_0}{s+\frac{Q}{V}+k_f[B]_0+k_r}\right)$$

Inverse Laplace: $\frac{1}{s+\alpha} \longleftrightarrow e^{-\alpha t}$

$$[A](t) = \frac{1}{k_f[B]_0+k_r}\left(k_r\left([A]_0+[C]_0\right)e^{-\frac{Q}{V}t} + \left(k_f[B]_0[A]_0+k_r[C]_0\right)e^{-\left(\frac{Q}{V}+k_f[B]_0+k_r\right)t}\right)$$

Similarly:

$$[C](s) = \frac{\left(s+\frac{Q}{V}\right)[C]_0 + k_f[B]_0\left([A]_0+[C]_0\right)}{\left(s+\frac{Q}{V}\right)\left(s+\frac{Q}{V}+k_f[B]_0+k_r\right)}$$

$$= \frac{1}{k_f[B]_0+k_r}\left(\frac{k_f[B]_0\left([A]_0+[C]_0\right)}{s+\frac{Q}{V}} - \frac{k_f[B]_0[A]_0+k_r[C]_0}{s+\frac{Q}{V}+k_f[B]_0+k_r}\right)$$

$$[C](t) = \frac{1}{k_f[B]_0+k_r}\left(k_f[B]_0\left([A]_0+[C]_0\right)e^{-\frac{Q}{V}t} - \left(k_f[B]_0[A]_0+k_r[C]_0\right)e^{-\left(\frac{Q}{V}+k_f[B]_0+k_r\right)t}\right)$$

2. **[40 pts]** Consider the following set of ODEs describing the dynamics of a biomechanical system with mass m and damping γ, with force $f(t)$ driving the input, and with velocity $v(t)$ at the output:

$$\begin{aligned} \frac{du}{dt} &= v(t) \\ m\,\frac{dv}{dt} &= -\gamma\, v(t) + f(t)\,. \end{aligned}$$

(a) [10 pts] Find the Laplace transform of velocity $v(s)$ as a function of the Laplace transform of the force $f(s)$, and initial value of velocity $v(0) = v_0$. Does it depend on the initial value of position $u(0) = u_0$, and why?

$$m\left(s\,v(s) - v_0\right) = -\gamma\, v(s) + f(s)$$

$$\left(ms + \gamma\right) v(s) = m v_0 + f(s)$$

$$v(s) = \frac{m v_0 + f(s)}{ms + \gamma}$$

Independent of initial position u(0) because without stiffness,
velocity v does not depend on the state of position u.

(b) [10 pts] For zero force $f(t) \equiv 0$, and for given initial conditions $u(0) = u_0$ and $v(0) = v_0$, find the velocity $v(t)$ as a function of time. Explain what you find.

$$v(s) = \frac{m v_0}{ms + \gamma} = \frac{v_0}{s + \frac{\gamma}{m}}$$

Inverse Laplace: $$\frac{1}{s+a} \longleftrightarrow e^{-at}$$

$$v(t) = v_0 \, e^{-\frac{\gamma}{m} t}$$

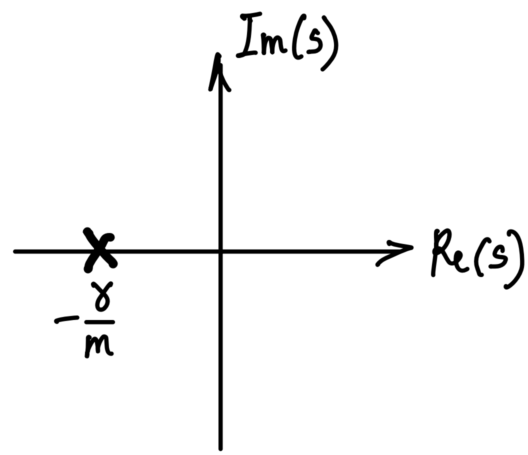
The velocity v decays exponentially, from its initial value, at a rate given by damping over mass.

(c) [5 pts] Find the transfer function $H(s) = v(s)/f(s)$ of the system, and find the poles and zeros.

$$H(s) = \frac{v(s)}{f(s)} = \frac{1}{ms + \gamma}$$

No zeros

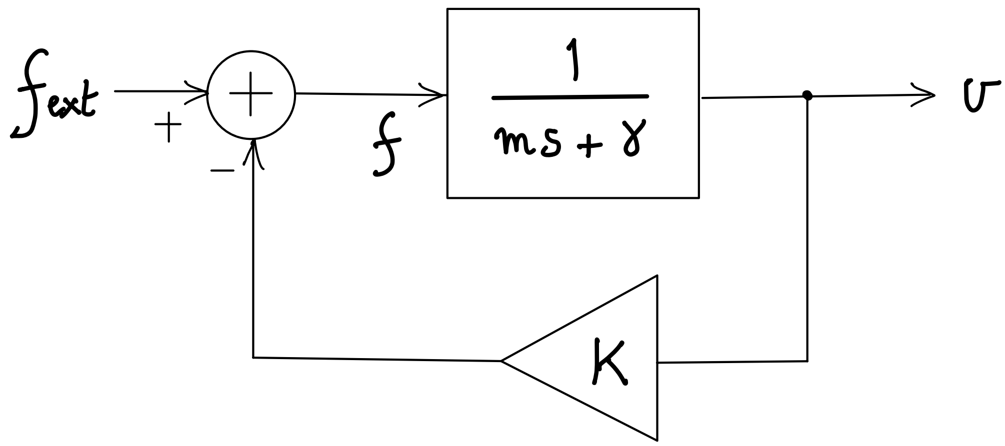
One pole: $s = -\frac{\gamma}{m}$



(d) [10 pts] Now consider closed-loop feedback, in which the force $f(t)$ is given by

$$f(t) = f_{ext}(t) - K\, v(t)$$

where $f_{ext}(t)$ is the externally applied force, and K is the feedback gain. Draw the closed-loop system block diagram, and find the closed-loop transfer function $F(s) = v(s)/f_{ext}(s)$.



$$v(s) = \frac{1}{ms+\gamma} f(s) = \frac{1}{ms+\gamma}\left(f_{ext}(s) - K v(s)\right)$$

$$(ms+\gamma)\, v(s) = f_{ext}(s) - K v(s)$$

$$(ms+\gamma+K)\, v(s) = f_{ext}(s)$$

$$F(s) = \frac{v(s)}{f_{ext}(s)} = \frac{1}{ms + \underbrace{(\gamma+K)}_{\text{Effective damping}}}$$

(e) [5 pts] Find the range of values for the feedback gain K in the closed-loop system to stabilize the biomechanical system with negative damping, $\gamma < 0$.

Stable for positive effective damping:

$$\gamma + K \geqslant 0$$

$$K \geqslant -\gamma$$

Range for K: $[-\gamma, +\infty[$

3. **[20 pts]** Linear time invariant and conservative biosystems:

(a) [5 pts] List various factors that cause blood pressure to vary across the body. Explain.

Resistance causes pressure to drop along the vasculature proportional to flow rate.

Potential energy in gravitation causes pressure to drop with increasing height.

Kinetic energy in velocity causes pressure to drop with increasing vessel diameter.

Etc.

(b) [5 pts] What determines the time constant of decay in the concentration of a compound in the blood stream? Explain.

The time constant is given by volume divided by flow rate.

This is because the rate of decay in the concentration is proportional to the rate at which the compound exits the volume through outward flow, and inversely proportional to that volume.

(c) [5 pts] How are the impulse response and the step response of a linear time invariant system related? Explain.

The step response is the time integral of the impulse response.

Conversely, the impulse response is the time derivative of the step response.

This is because the step function is the integral of the delta Dirac impulse function, integration is a linear operator, and the response is linear in the input.

(d) [5 pts] Explain the equivalence between initial conditions in the output of a linear time invariant system with zero input, and impulse activation ("all at once") of its input at time zero with zero initial conditions.

An impulse activation of the input at time zero produces a step increment in the output at time zero. Hence the output response for an impulse input at time zero from zero initial output at time just before zero, is equivalent to the output response of the same system from an initial output equal to that step increment immediately after time zero.

Specifically:

ODE: $$\frac{du}{dt} = A\, f(t) + \underbrace{\cdots}_{\text{linear terms in u and the other state variables}}$$

$$= A\, \delta(t) + \underbrace{\cdots}_{\text{nothing delta here!}}$$

$$\Rightarrow \quad \underbrace{u(0^+)}_{\text{just after time zero}} = A + \underbrace{u(0^-)}_{\text{just before time zero}}$$