BEGN 122A Fall 2020

Quiz 2

Tuesday, November 24, 2020

Name *(Last, First)*: SOLUTIONS

- This quiz is open book, open notes, and online, but web search is prohibited. You may follow electronic links from Canvas or the class web pages, but not any further. **No collaboration or communication in any form is allowed**, except for questions to the instructor and TAs.
- The quiz is due November 25, 2020 at 11:59pm, over Canvas. It should approximately take 2 hours to complete, but there is no time limit other than the submission deadline. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.
- There are 3 problems. Points for each problem are given in **[brackets]**. There are 100 points total.

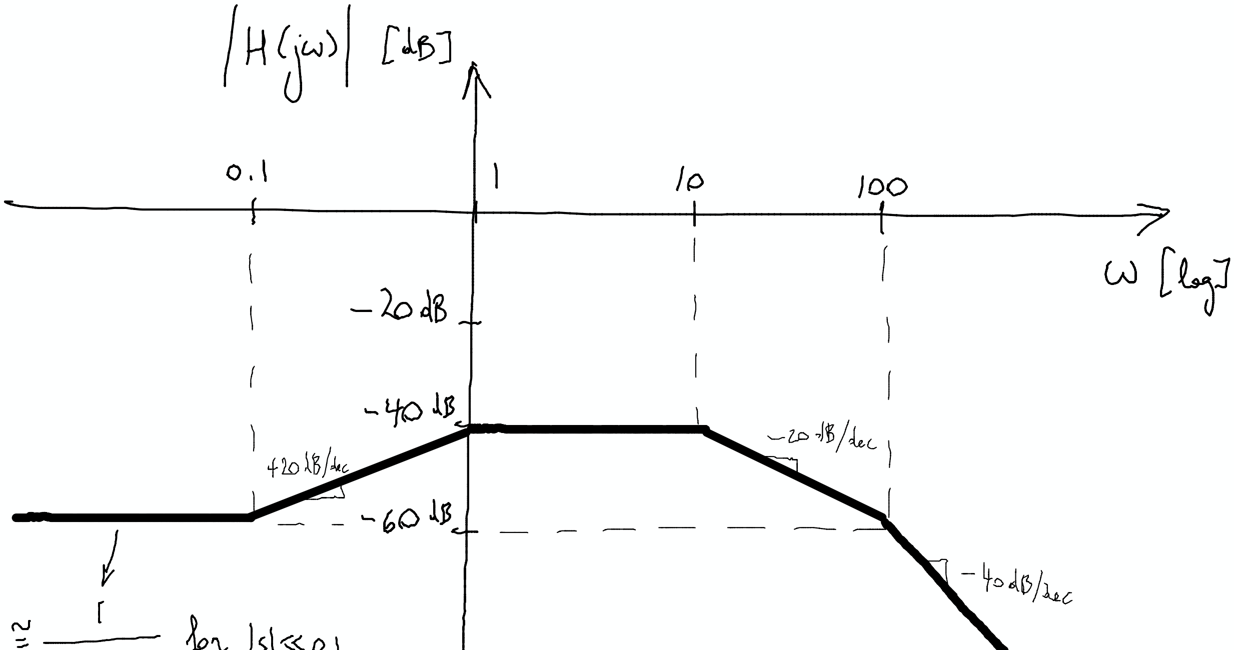
1. **[50 pts]** Consider the following linear time-invariant (LTI) biosystem:

$$H(s) = \frac{10s+1}{(s+1)(s+10)(s+100)}.$$

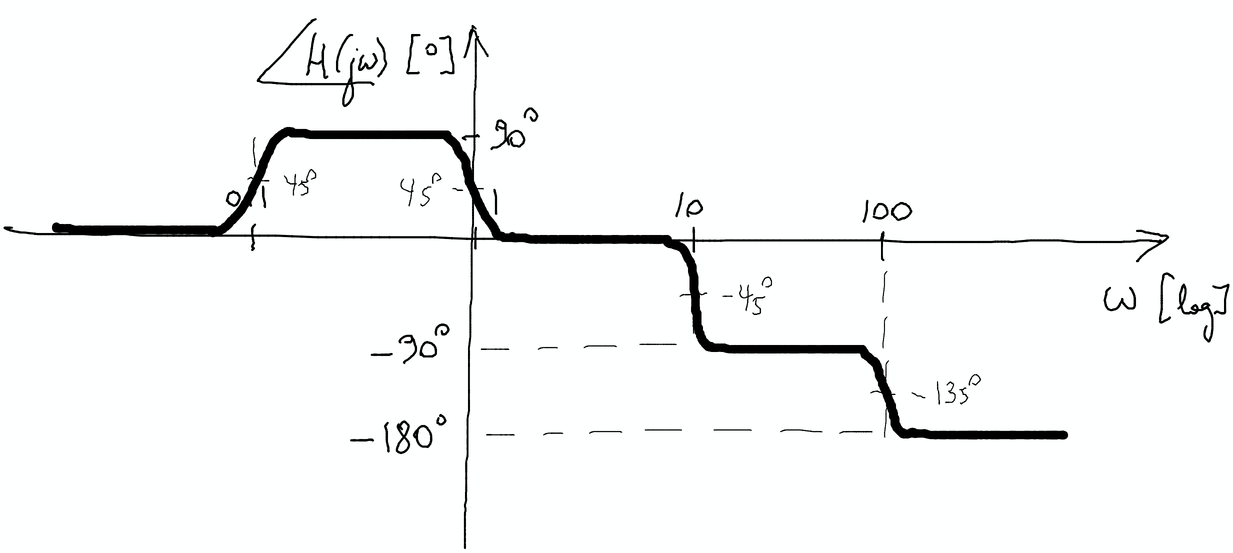
(a) [10 pts] Sketch the Bode plot.

Zero @ $s = -0.1$

Poles @ $s = \begin{cases} -1 \\ -10 \\ -100 \end{cases}$



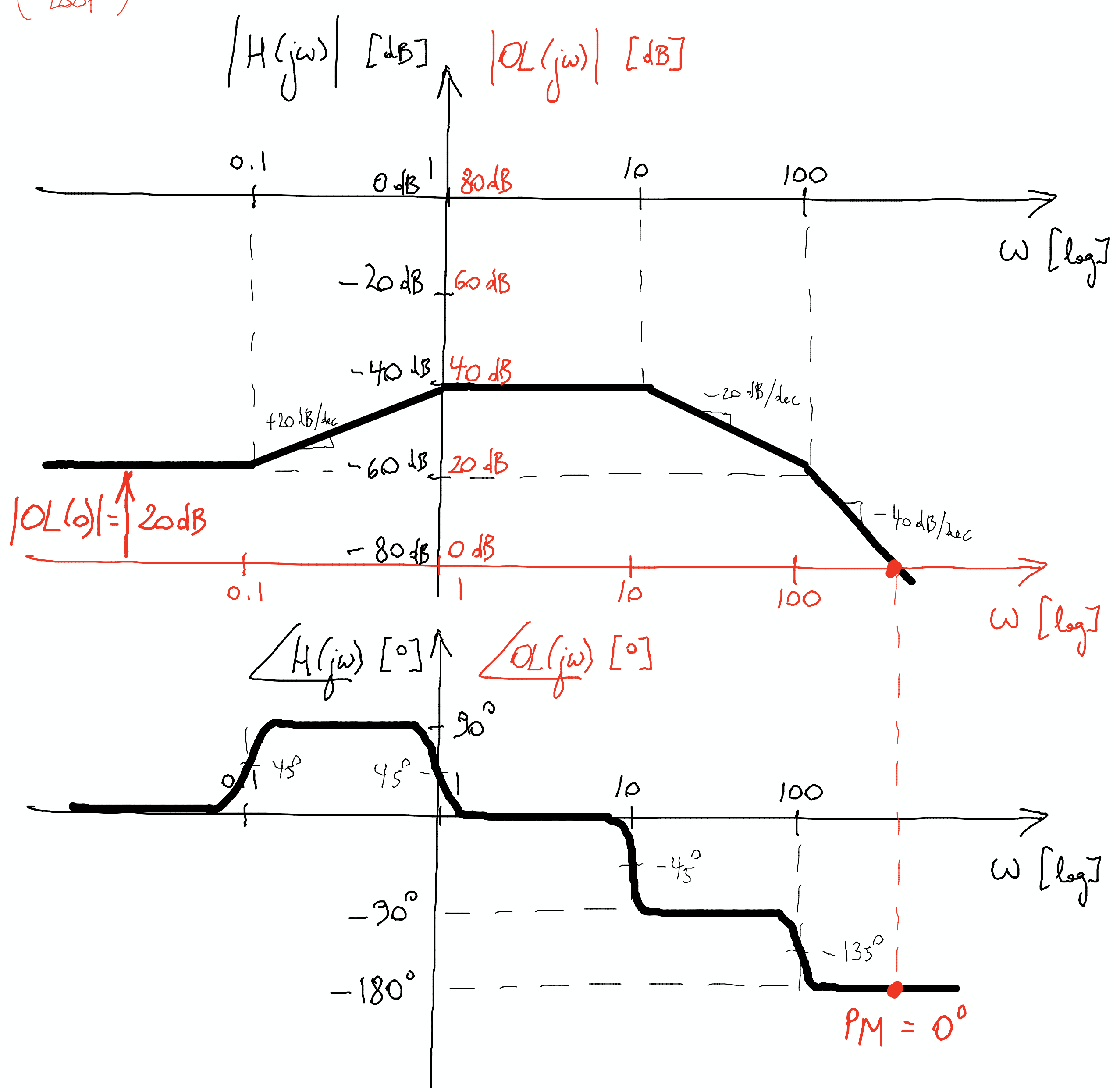
$H(s) \cong \frac{1}{1 \cdot 10 \cdot 100}$ for $|s| \ll 0.1$



(b) [10 pts] Find the closed-loop DC error, and phase margin, for proportional control with 80 dB gain.

$OL(j\omega) = K_p \cdot H(j\omega)$ with $K_p = 10{,}000$ (80 dB)

(OPEN LOOP)



Closed-loop DC error: $\left| 1 - CL(0) \right| = \left| 1 - \frac{OL(0)}{1+OL(0)} \right| = \left| \frac{1}{1+OL(0)} \right| = \frac{1}{11} = 9\%$

Phase margin: $\angle OL(j\omega) = -180^\circ$ where $|OL(j\omega)| = 0$ dB $\Rightarrow$ $PM = 0^\circ$

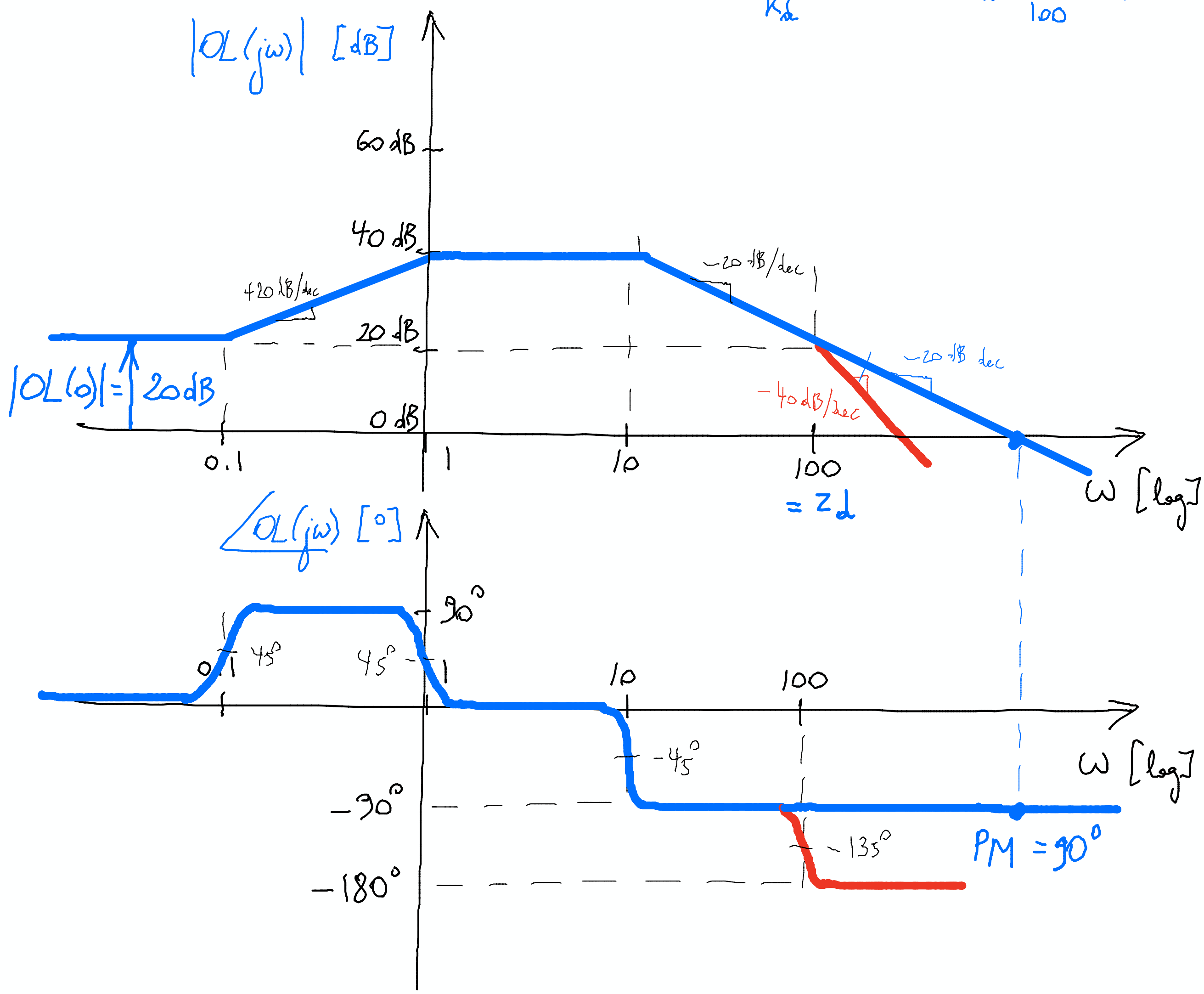
(c) [10 pts] Now add derivative control, keeping the proportional control at the same level. Find the minimum value of derivative gain to give a phase margin at least 90°. Find the closed-loop DC error.

$$OL(j\omega) = (K_p + K_d j\omega) \cdot H(j\omega) \quad \text{with} \quad K_p = 10{,}000 \ (80\,dB), \quad K_d = 100 \ (40\,dB)$$

PD zero canceling pole @ -100: (to increase PM by 90°)

$K_p + K_d s = 0$ @ $s = -z_d = -100$

$\Rightarrow z_d = \frac{K_p}{K_d} = 100$ or $K_d = \frac{K_p}{100} = 100$



Closed-loop DC error: $|1 - CL(0)| = \left|1 - \frac{OL(0)}{1 + OL(0)}\right| = \left|\frac{1}{1 + OL(0)}\right| = \frac{1}{11} = 9\%$

(same!)

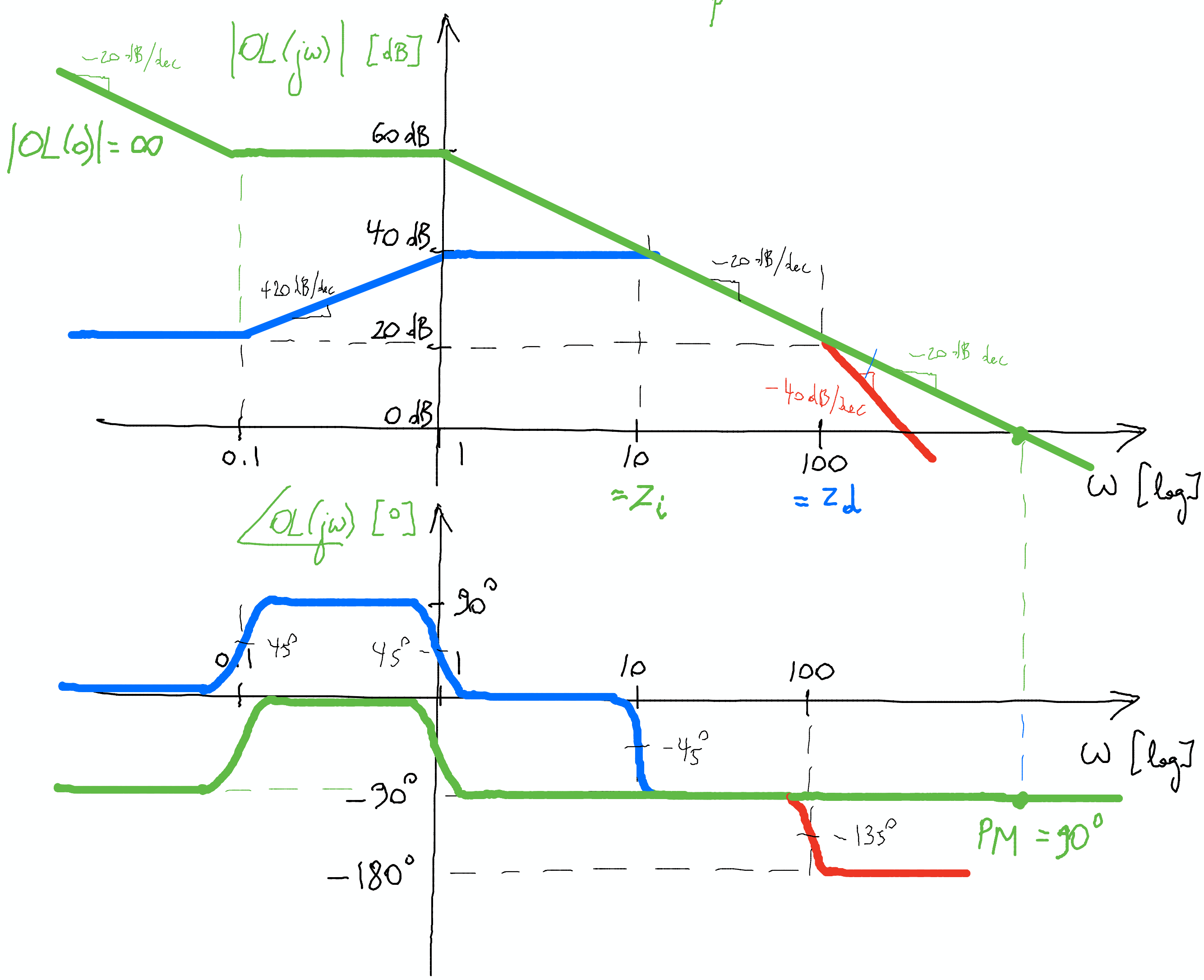
(d) [10 pts] Now add integral control, keeping proportional and derivative control at the same levels. Find the maximum value of integral gain to maintain phase margin at least 90°. Find the closed-loop DC error.

$$OL(j\omega) = \left(K_p + K_d j\omega + \frac{K_i}{j\omega}\right) \cdot H(j\omega) \quad \text{with} \quad \begin{array}{ll} K_p = 10{,}000 & (80\text{ dB}) \\ K_d = 100 & (40\text{ dB}) \\ K_i = 100{,}000 & (100\text{ dB}) \end{array}$$

PI zero canceling pole @ -10:
(to increase DC error while
maintaining PM)

$$K_p + K_i \frac{1}{s} \cong 0 \quad @ \quad s = -z_i = -10$$

$$\Rightarrow z_i = \frac{K_i}{K_p} = 10 \quad \text{or} \quad K_i = 10 K_p = 100{,}000$$

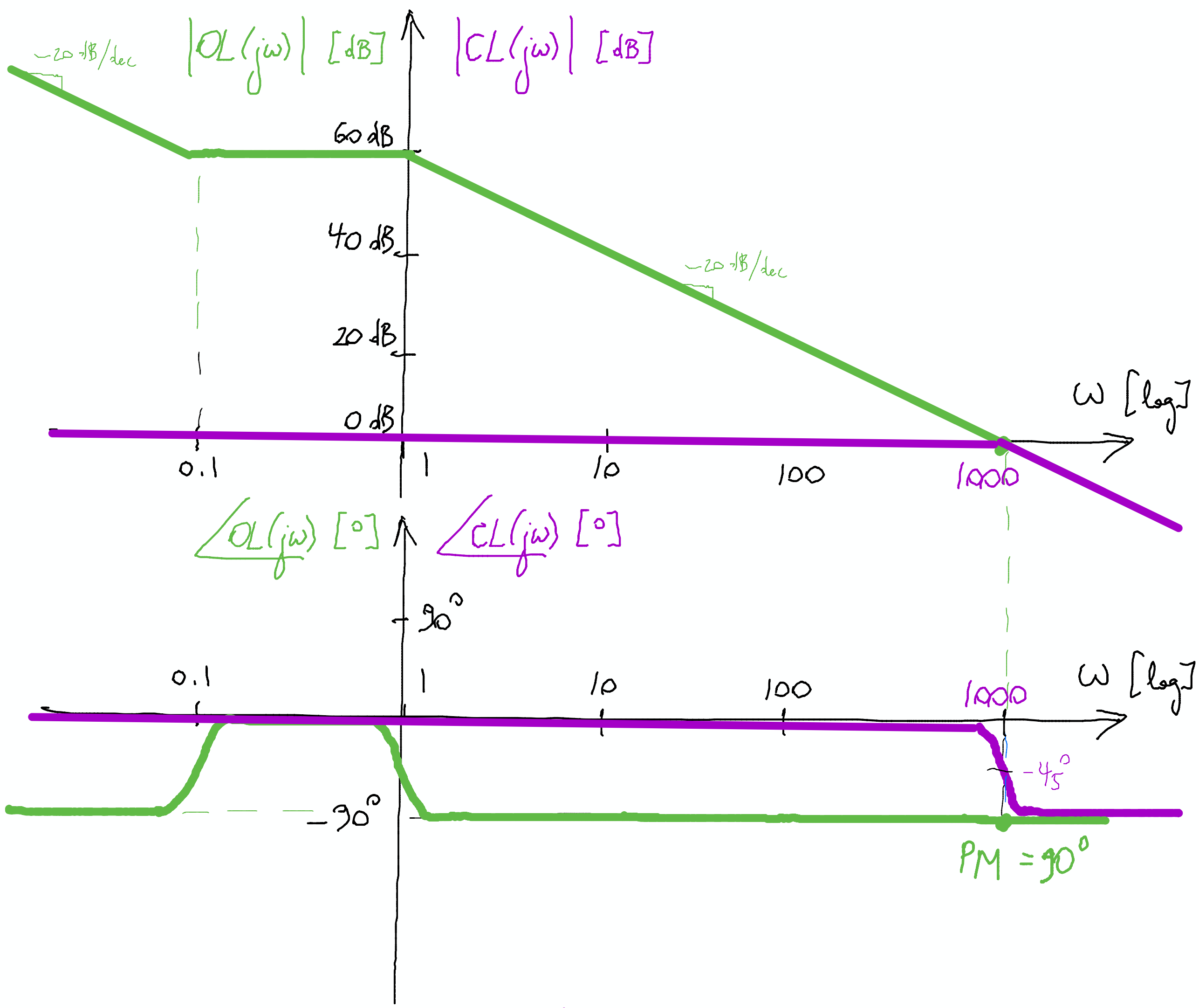


Closed-loop DC error: $$|1 - CL(0)| = \left|1 - \frac{OL(0)}{1 + OL(0)}\right| = \left|\frac{1}{1 + OL(0)}\right| = \frac{1}{\infty} = 0$$

(e) [10 pts] Find the closed-loop transfer function for these values of proportional, integral, and derivative gain, and sketch the Bode plot. Validate that the closed-loop DC error and high-frequency dynamics are consistent with those predicted by the above open-loop analysis.

$$CL(s) = \frac{OL(s)}{1 + OL(s)} \approx \begin{cases} 1 & \text{for } |OL(s)| \gg 1 \\ OL(s) & \text{for } |OL(s)| \ll 1 \end{cases}$$

More precisely: (after pole cancelations) $OL(s) \approx \dfrac{100(10s+1)}{s(s+1)} \Rightarrow CL(s) \approx \dfrac{1000s + 100}{s^2 + 1001s + 100}$



Closed-loop DC error: $\left|1 - CL(0)\right| = \left|1 - 1\right| = 0$

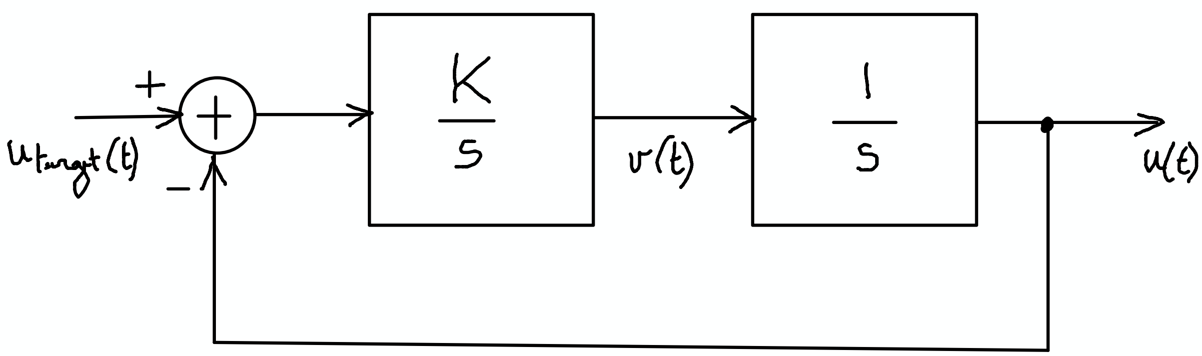
High-frequency dynamics: Approximately single-pole lowpass, consistent with 90 deg phase margin at the cut-off radial frequency 1,000.

2. **[30 pts]** Here we consider the dynamics of a coupled set of two ordinary differential equations describing a biomechanical control system in state variables $u(t)$ and $v(t)$, and with target u_{target} for $u(t)$:

$$\begin{aligned}\frac{du}{dt} &= v(t)\\ \frac{dv}{dt} &= K\left(u_{\text{target}} - u(t)\right)\end{aligned}$$

where K is a constant.

(a) [10 pts] Show a block diagram of this closed-loop system interconnecting two blocks: a "controller" generating v from u and its target u_{target}; and a "biosystem" generating u from v. You may assume that there is no measurement error in any of the state variables. What is the controller type?



Integral control with gain K

(b) [5 pts] Find the open-loop transfer function, and find the phase margin. What does it imply about the stability of the closed-loop system?

$$OL(s) = \frac{K}{s} \cdot \frac{1}{s} = \frac{K}{s^2}$$

The phase is everywhere -180 deg, so the phase margin is 0.

Hence, the system is critically (marginally) stable.

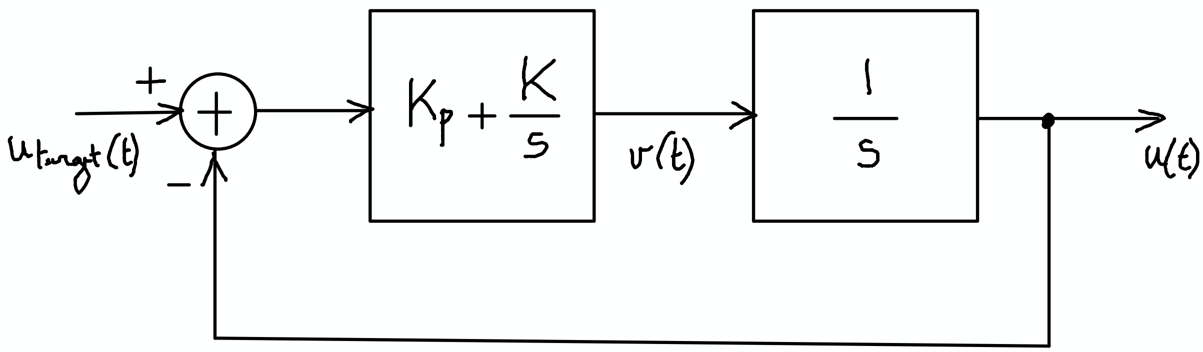
(c) [5 pts] Find the closed-loop transfer function, and find the poles. Check the consistency of the stability of the closed-loop dynamics with the above open-loop analysis.

$$CL(s) = \frac{OL(s)}{1+OL(s)} = \frac{\frac{K}{s^2}}{1+\frac{K}{s^2}} = \frac{1}{1+\frac{s^2}{K}}$$

Poles: $s = p_{1,2} = \pm j\sqrt{K}$ on the imaginary axis

Hence, critically (marginally) stable

(d) [10 pts] Now consider the effect of additional proportional control on the stability of the closed-loop system. Find the value of the proportional gain to produce critically damped second-order dynamics in the closed-loop response.



$$OL(s) = \left(K_p + \frac{K}{s}\right)\frac{1}{s} = \frac{K_p s + K}{s^2}$$

$$CL(s) = \frac{OL(s)}{1 + OL(s)} = \frac{\frac{K_p s + K}{s^2}}{1 + \frac{K_p s + K}{s^2}} = \frac{K_p s + K}{s^2 + K_p s + K}$$

Critically damped for $K_p = 2\sqrt{K}$

3. **[20 pts]** Consider any system described by a transfer function given by the ratio of two polynomials $H(s) = N(s)\,/\,D(s)$, with numerator $N(s)$, and denominator $D(s)$. A surprisingly simple criterion on the stability of such system is in terms of the polarity of the coefficients a_i of the denominator polynomial,

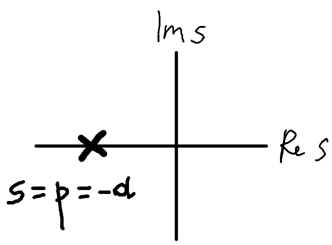
$$D(s) = s^n + a_{n-1}\,s^{n-1} + \ldots + a_1\,s + a_0\,.$$

   (a) [10 pts] Show that if the system $H(s)$ is (strictly) stable, then all denominator coefficients a_i must be (strictly) positive. In other words, any system with at least one non-positive (negative) denominator coefficient is surely (strictly) unstable.

   Hint: Realize that $D(s)$ is given by a product of terms $s - p_i$ for each of the poles p_i, where any complex poles come in conjugate pairs. Show that for a stable system each real pole contributes a first-order section with positive coefficient ($s + a$ with $a \geq 0$), and each complex conjugate pair of poles contributes a second-order section with all positive coefficients ($s^2 + a\,s + b$ with $a \geq 0$ and $b \geq 0$).

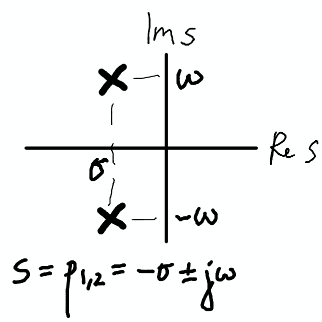
For a (strictly) stable system, all poles have real parts that are (strictly) positive.

Each real pole contributes a first-order section with (strictly) positive coefficients:



$$(s - p) = (s + a) \quad \text{with} \quad a \underset{(>)}{\geq} 0$$

Each pair of complex conjugate poles contributes a second-order section with (strictly) positive coefficients:



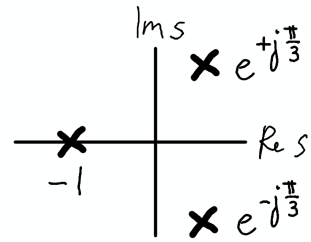
$$(s - p_1)(s - p_2) = (s + \sigma - j\omega)(s + \sigma + j\omega)$$

$$= s^2 + 2\sigma\, s + (\sigma^2 + \omega^2) \quad \text{with} \quad 2\sigma \underset{(>)}{\geq} 0, \quad \sigma^2 + \omega^2 \underset{(>)}{\geq} 0$$

The denominator $D(s)$ is the product of all these sections, each being a polynomial with (strictly) positive coefficients. Hence, $D(s)$ has all (strictly) positive coefficients.

(b) [10 pts] Show that the converse is *not* true. Give a counter-example of a third-order system with all positive denominator coefficients that is unstable.

$$D(s) = s^3 + 1$$



$$s = p_{1,2,3} = \sqrt[3]{-1}$$

(c) Bonus [+10 extra pts] Show that a system $H(s)$ with all positive denominator coefficients can not have any positive real pole. In order words: any unstable system with all positive coefficients can only have complex conjugate poles in the right half plane. This implies that any instability in a system with only positive denominator coefficients must be oscillatory, either undamped or exponentially increasing in amplitude. How would you approximately find that frequency of oscillation, and that rate of exponential growth, from the open-loop Bode plot?

Any positive real pole contributes a first-order section with a negative coefficient, causing $D(s)$ to have at least one negative coefficient.

The gain margin, the open-loop gain at the frequency where the open-loop phase reaches +/- 180 degrees, determines the rate of exponential growth at that frequency of oscillation. See Practice Quiz 2 Problem 3 for a similar argument with phase margin.