BENG 122A Fall 2021

Quiz 2

Tuesday, November 16, 2021

Name *(Last, First):* SOLUTIONS

- This quiz is open book, open notes, and online, but web search is prohibited. You may follow electronic links from Canvas or the class web pages, but not any further. **No collaboration or communication in any form is allowed**, except for questions to the instructor and TAs.
- The quiz is due November 17, 2021 at 11:59pm, over Canvas. It should approximately take 2 hours to complete, but there is no time limit other than the submission deadline. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.
- There are 3 problems. Points for each problem are given in **[brackets]**. There are 100 points total.

1. **[50 pts]** Consider the following linear time-invariant (LTI) biosystem:

$$H(s) = \frac{10s}{(s+0.1)(s+1)(s+10)}.$$

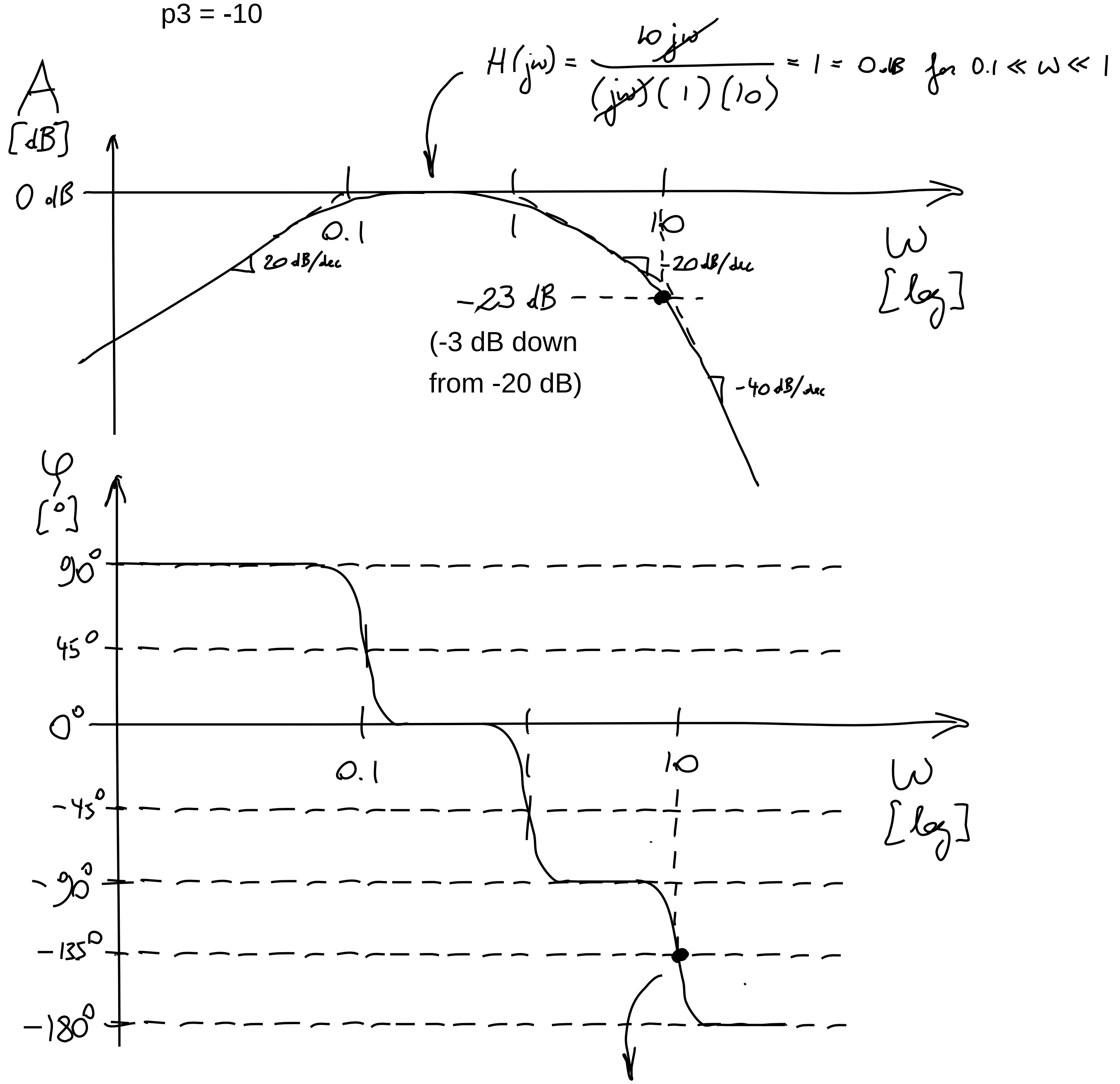
(a) [10 pts] Sketch the Bode plot.

Zero @ z = 0

Poles @ p1 = -0.1

p2 = -1

p3 = -10



45 degree phase margin is reached by raising the open-loop gain by Kp = 23 dB to 0 dB

(b) [10 pts] First consider purely proportional control. Find the largest value of proportional gain K_p for which the closed-loop system is stable with at least 45 degree phase margin. Find the closed-loop DC error.

See page 2: 45 degree phase margin is reached when the open-loop gain at ω = 10 is raised from -23 dB to 0 dB. Hence the maximum Kp is +23 dB.

The DC gain is 0 because of the zero at s = 0. Hence the closed-loop DC error is 1 (or 100%).

$$CL(0) = \frac{OL(0)}{1 + OL(0)} = 0$$

$$\text{error}(0) = 1 - CL(0) = \frac{1}{1+OL(0)} = 1$$

(c) [10 pts] Now add integral control. Find the values of integral gain K_i and proportional gain K_p that minimize the closed-loop DC error, while maintaining stable closed-loop response with at least 45 degree phase margin. Find the closed-loop DC error.

Controller $K_p + \frac{K_i}{s}$ adds: $\begin{cases} \text{A pole @ } s = 0 \\ \text{A zero @ } s = z_i = -\frac{K_i}{K_p} \end{cases}$

Strategy:

1) Maintain Kp at +23dB to maintain 45 degree phase margin. No higher Kp is possible without lowering the phase margin.

2) Have the controller zero z_i cancel the biosystem second pole p2. No higher Ki is possible without lowering the phase margin.

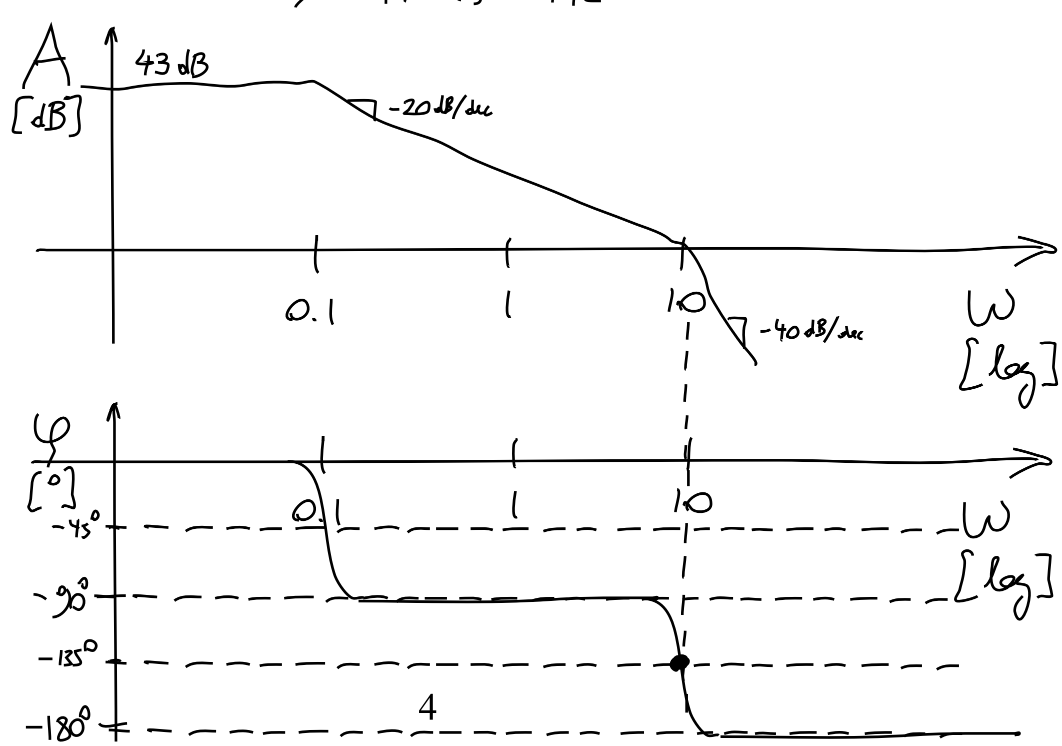
$$z_i = -\frac{K_i}{K_p} = -1 \Rightarrow K_i = K_p = 23\text{dB} = 14.1$$

3) The controller pole cancels the biosystem zero. Open-loop DC gain is

$$OL(0) = 10\,K_i = 20\text{dB} + 23\text{dB} = 43\text{ dB} = 141$$

and hence the closed-loop DC error is

$$\text{error}(0) = \frac{1}{1+OL(0)} = \frac{1}{142} = 0.7\,\%$$



(d) [10 pts] Now add derivative control, keeping proportional and integral gains at the same levels. Find the value of derivative gain K_d to maintain a phase margin of at least 90°. Find the closed-loop DC error.

Additional derivative control $K_d\, s$ adds a second zero @ s = $z_d \cong -\frac{K_p}{K_d}$

Strategy: Have the second controller zero z_d cancel the third biosystem pole p_3 so that phase margin becomes 90 degrees.

$$z_d \cong -\frac{K_p}{K_d} = -10 \quad \Rightarrow \quad K_d \cong 0.1\, K_d = 3\ \text{dB} = 1.41$$

Closed-loop DC error remains the same: 0.7%.

Only one pole p_1 remains for a first-order lowpass open-loop response:

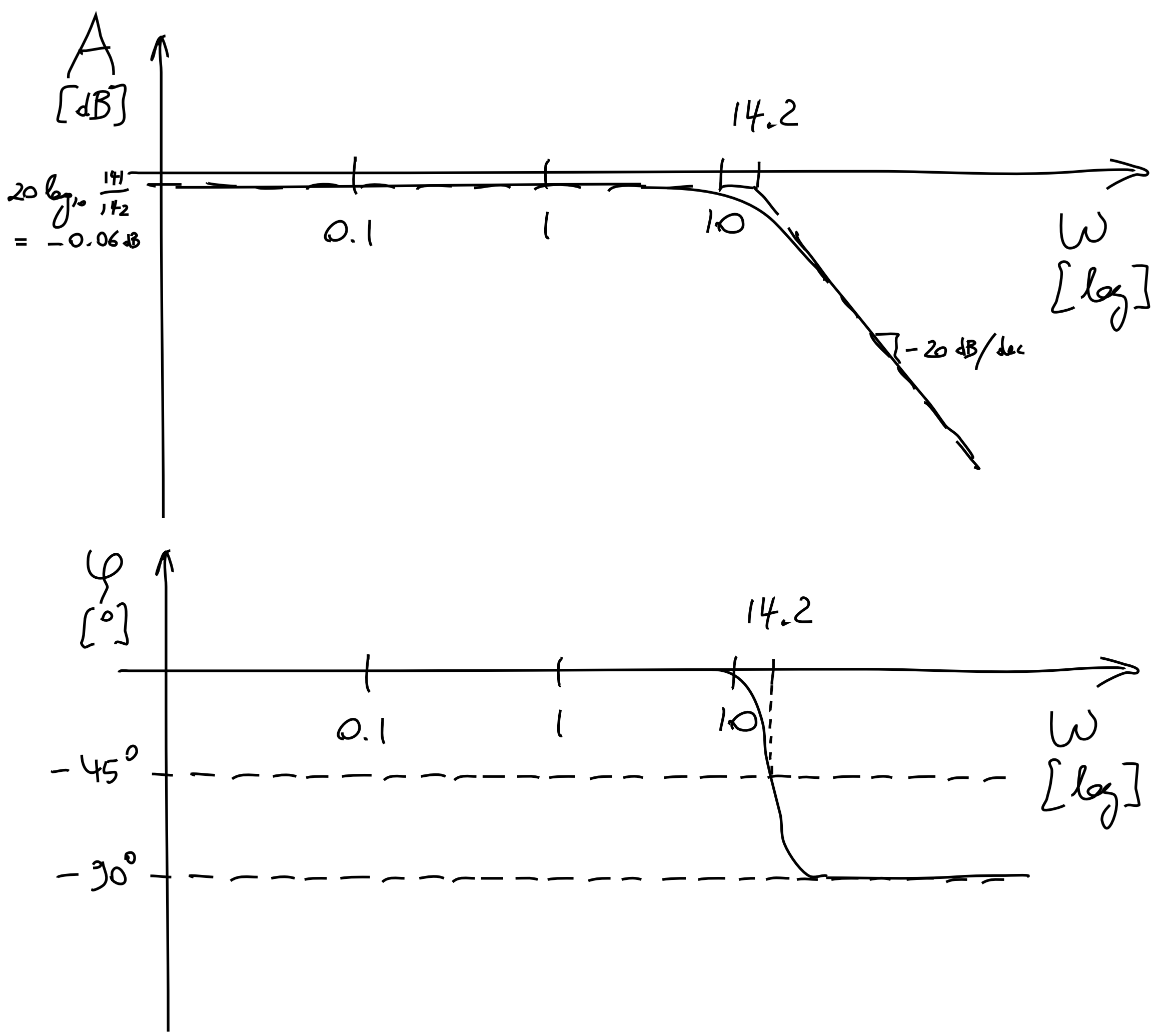
$$OL(s) \cong \frac{141}{1 + 10s}$$

(e) [10 pts] Find the closed-loop transfer function for these values of proportional, integral, and derivative gain, and sketch the Bode plot. Validate that the closed-loop DC error and high-frequency dynamics are consistent with those predicted by the above open-loop analysis.

$$CL(s) = \frac{OL(s)}{1 + OL(s)} \cong \frac{141}{142 + 10s}$$ from (d)

This first-order lowpass closed-loop response with DC gain 141/142 is consistent with the 90 degree phase margin and 0.7% DC error predicted by the open-loop analysis in (d).

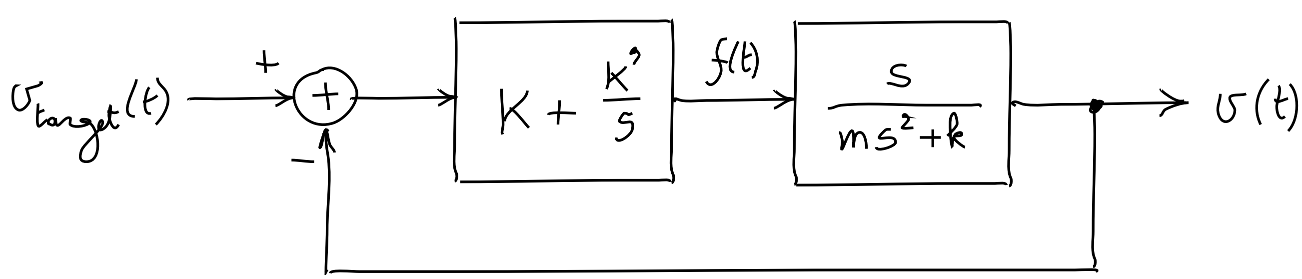
Closed-loop Bode plot:



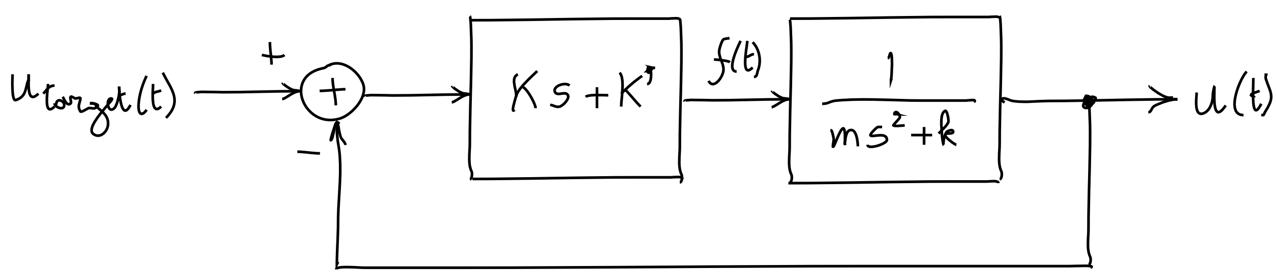
2. **[30 pts]** Here we consider the dynamics of a coupled set of two ordinary differential equations in state variables $u(t)$ and $v(t)$ describing a mechanical biosystem with mass m and stiffness k driven by a force input $f(t)$:

$$\begin{aligned} \frac{du}{dt} &= v(t) \\ m\,\frac{dv}{dt} &= -k\,u(t) + f(t). \end{aligned}$$

(a) [10 pts] Show a block diagram of a closed-loop system with a proportional-integral controller driving the velocity $v(t)$ of the biosystem towards a target $v_{\text{target}}(t)$ with proportional gain K and integral gain K'. You may assume that there is no measurement error in the output $v(t)$.



(b) [10 pts] Now consider a closed-loop system with a proportional-derivative controller driving the position $u(t)$ of the biosystem towards a target $u_{\text{target}}(t)$ with proportional gain K' and derivative gain K, again without measurement error. Show that the two closed-loop systems are equivalent, generating the same state variables $u(t)$ and $v(t)$, under the condition $u_{\text{target}}(t) = \int v_{\text{target}}(t)dt$.



Both systems have identical open-loop transfer functions:

$$\left(K + \frac{K'}{s}\right) \frac{s}{ms^2+k} = \left(Ks + K'\right) \frac{1}{ms^2+k}$$

so they have also identical closed-loop transfer functions.

If a target $u_{target}(t)$ generates a response $u(t)$

then a target $\frac{d}{dt} u_{target}(t)$ must generate a response $\frac{d}{dt} u(t)$

hence a target $v_{target}(t)$ generates a response $v(t)$

and vice versa.

(c) [10 pts] For mass $m = 2$ kg and stiffness $k = 1\ \mathrm{kg\,s^{-2}}$, find the control gain parameters K and K' to produce a critically damped closed-loop response with time constant $\tau = 1$ s (two coinciding poles at $-1\ \mathrm{s^{-1}}$). What is the closed-loop DC error, and how could you improve on it?

Closed-loop transfer function:

$$CL(s) = \frac{OL(s)}{1 + OL(s)} = \frac{Ks + K'}{ms^2 + Ks + (K' + k)}$$

A critically damped response with time constant τ requires a denominator of the form:

$$m\left(s + \frac{1}{\tau}\right)^2 = ms^2 + \frac{2m}{\tau}s + \frac{m}{\tau^2}$$

and hence:

$$K = \frac{2m}{\tau} = 4\ \frac{\mathrm{kg}}{\mathrm{s}}$$

$$K' = \frac{m}{\tau^2} - k = 2\ \frac{\mathrm{kg}}{\mathrm{s^2}} - 1\ \frac{\mathrm{kg}}{\mathrm{s^2}} = 1\ \frac{\mathrm{kg}}{\mathrm{s^2}}$$

Closed-loop DC error:

$$1 - CL(0) = 1 - \frac{K'}{K' + k} = \frac{k}{K' + k} = \frac{1}{2}$$

can be made zero by adding integral control to the proportional-derivative position controller.

3. **[20 pts]** Short answer questions– let your imagination and creativity go loose!

   (a) [10 pts] Give an example of an unstable biosystem that is stabilized by proportional feedback control.

   Any first-order system with strictly positive growth rate, e.g. population dynamics of a single species reproducing at a constant rate, such as yeast under unlimited food supply.

   Proportional control with gain greater than the growth rate stabilizes the system, reversing the growth to control the population towards zero. For yeast this could be an antibiotic that kills yeast at a rate greater than its growth.

   (b) [10 pts] Give an example of a stable biosystem that becomes unstable under the effect of measurement delay in proportional feedback control.

   Any second-order system that has a phase margin, in the absence of measurement delay, already strictly less than 90 degrees. Then any measurement delay that produces a phase lag greater than that phase margin at its frequency, results in a negative phase margin and thus an unstable response.

   An example could be a scalpel arm for robot-assisted surgery with mass, stiffness and damping. The controller drives the arm to steer its coordinates towards a target location in the tissue. With purely proportional control, measurement delay in the arm position may lead to instability, which can be rectified by adding derivative control.