BEng 122A Fall 2022

Quiz 2

Tuesday, November 22, 2022

Name *(Last, First)*: SOLUTIONS

- This quiz is open book, open notes, and online, but web search is prohibited. You may follow electronic links from Canvas or the class web pages, but not any further. **No collaboration or communication in any form is allowed**, except for questions to the instructor and TAs.
- The quiz is due November 23, 2022 at 11:59pm, over Canvas. It should approximately take 2 hours to complete, but there is no time limit other than the submission deadline. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.
- There are 3 problems. Points for each problem are given in **[brackets]**. There are 100 points total.

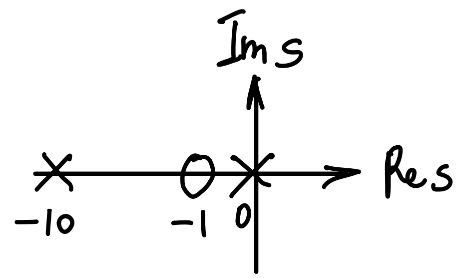
1. **[50 pts]** Consider the following linear time-invariant (LTI) biosystem:

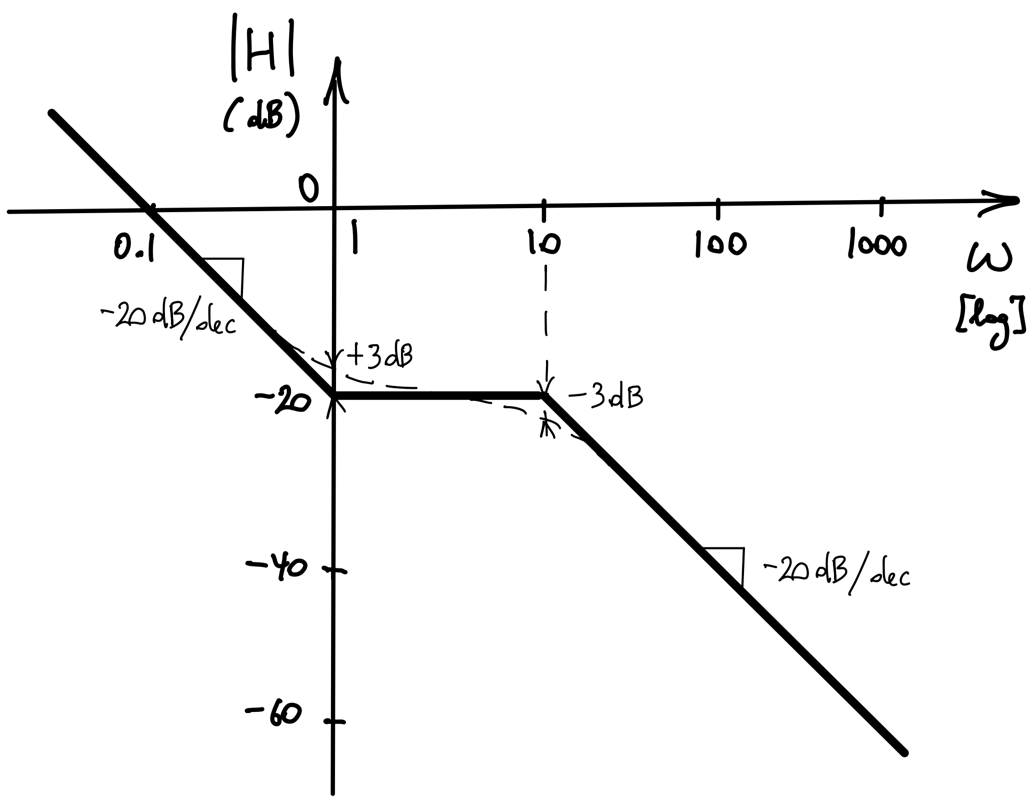
$$H(s) = \frac{s+1}{s^2+10s}. \quad = \quad \frac{s+1}{s(s+10)}$$

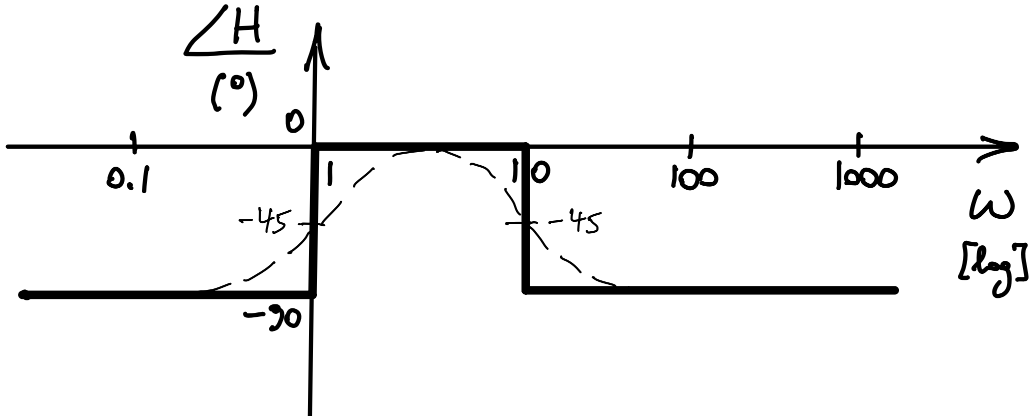
(a) [10 pts] Sketch the Bode plot.

Zero @ $s = -1$

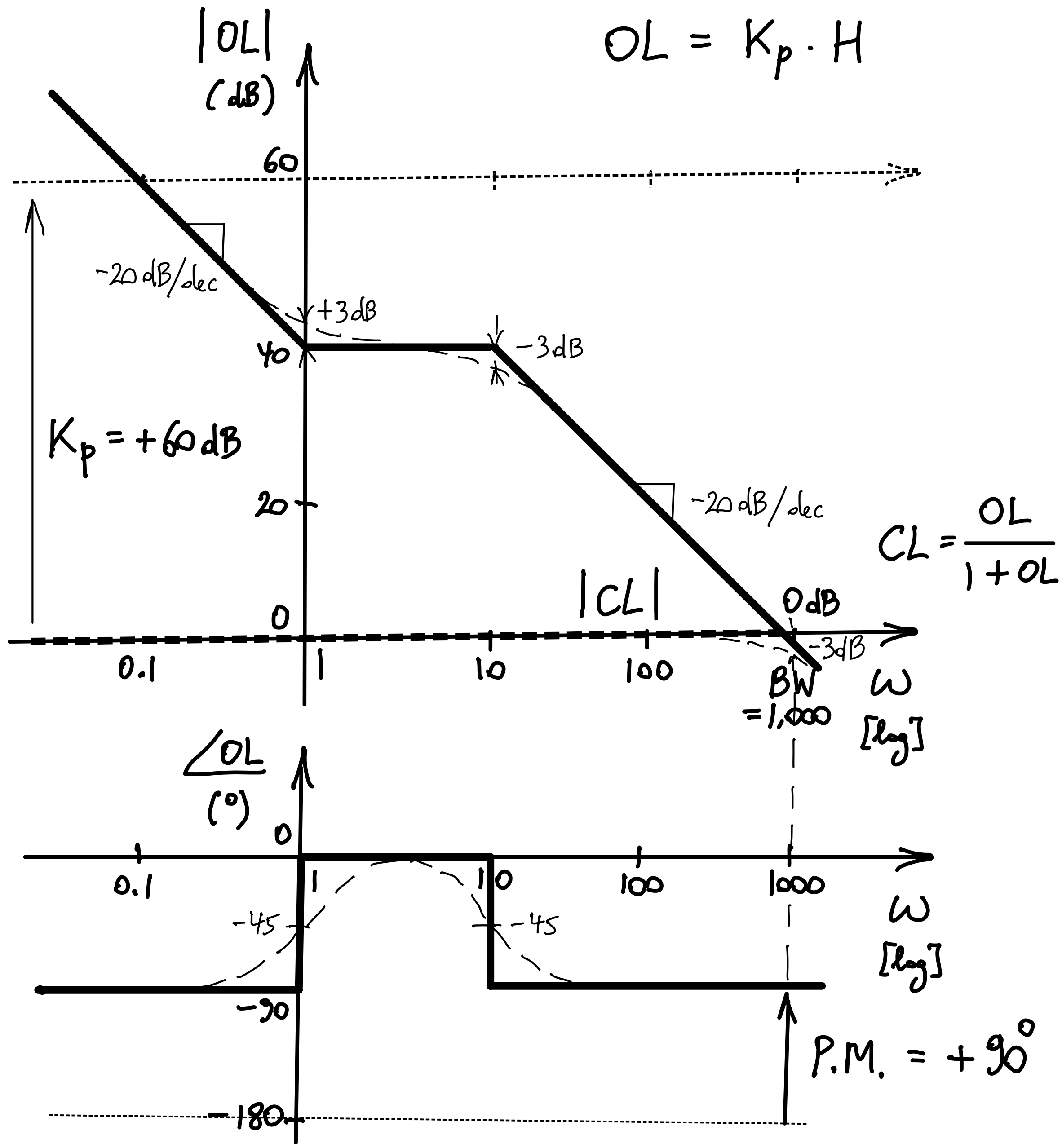
Poles @ $s = 0, -10$







(b) [10 pts] First consider proportional control with 60 dB gain ($K_p = 1{,}000$), without measurement error. Find the phase margin of the open-loop system, and find the DC error and the -3 dB bandwidth of the closed-loop system.



DC error = 0 $\quad \left(OL(0) = \infty, \text{ so } CL(0) = 1 \right)$

Phase margin = $+90^\circ$ $\quad \left(|OL| = 1 \text{ and } \angle OL = -90^\circ \text{ @ } \omega = 1000 \right)$

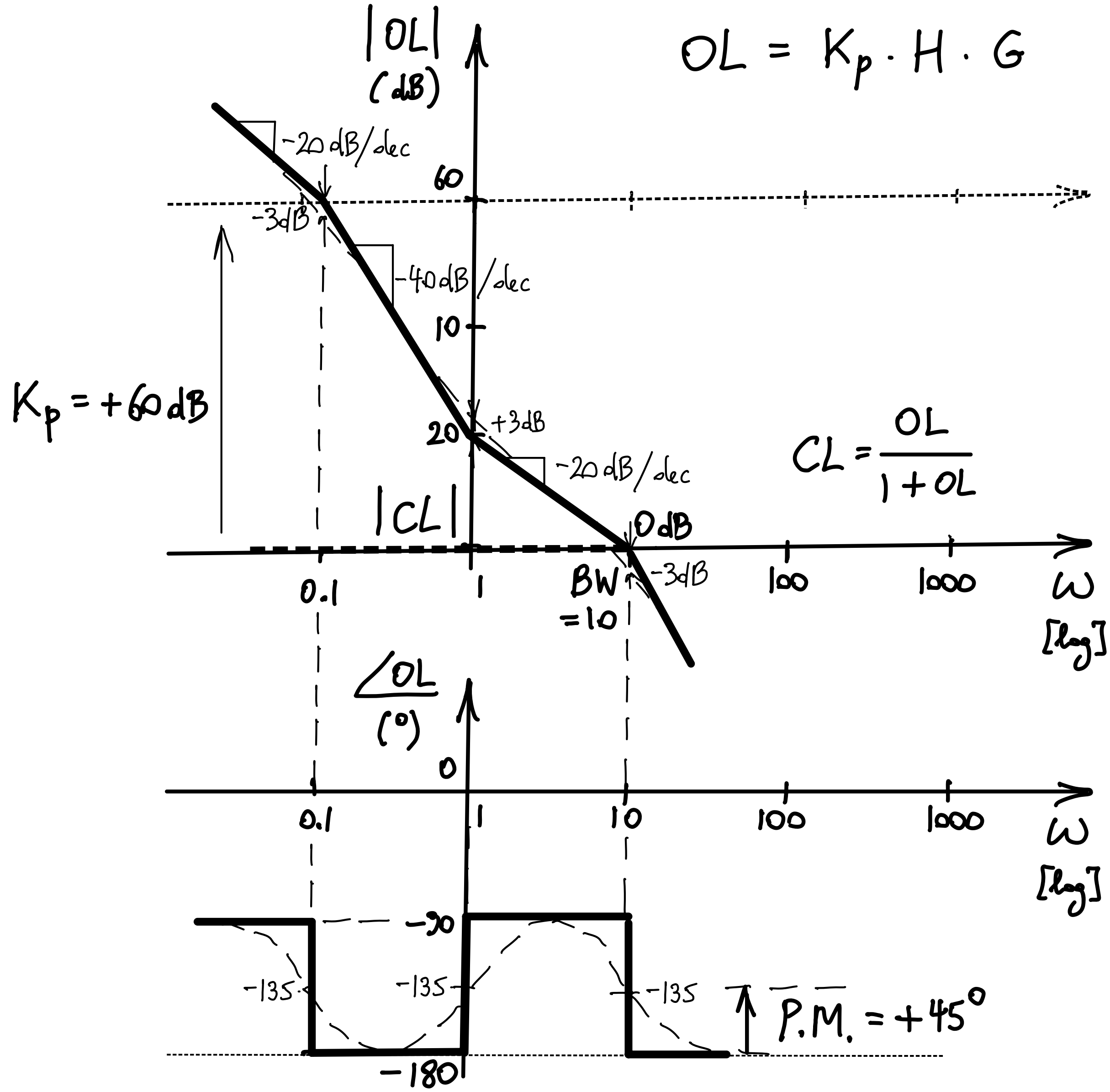
Closed-loop −3dB BW = 1,000 $\quad \left(|CL| = \dfrac{|OL|}{\sqrt{1 + |OL|^2}} = \dfrac{1}{\sqrt{2}} \text{ @ } \omega = 1000 \right)$

(c) [10 pts] Now consider error in the measurement of the biosystem, with the measurement system given by:

$$G(s) = \frac{1}{1 + 10s}.$$

Extra -20dB/dec magnitude and -90deg phase for frequencies > 0.1

Find the maximum value of the proportional gain K_p maintaining stable closed-loop response with at least 45° phase margin. How does this affect the DC error and the -3 dB bandwidth of the closed-loop system? Explain.



DC error = 0

Unaffected.

Closed-loop -3dB BW = 10

Worse -3dB bandwidth due to the extra delay in the feedback loop.

(d) [10 pts] Consider the biosystem again with measurement error, but now with proportional-derivative control with 60 dB proportional gain ($K_p = 1{,}000$). Find the minimum value for the derivative gain K_d to obtain a phase margin of at least 90°. Show that this design achieves the same DC error and -3 dB bandwidth of the closed-loop system without measurement error in (b).

$$OL = (K_p + K_d s) \cdot H \cdot G$$

$$= K_p \cancel{\left(1 + \frac{K_d}{K_p} s\right)} \cdot H \cdot \cancel{\frac{1}{1 + 10 s}}$$

Cancel measurement pole with PD zero for:

$$K_d = 10\ K_p = 10{,}000$$

This lifts the phase by 90 deg, for a 90 deg phase margin.

Lower derivative gain would reduce that phase margin.

The resulting open-loop transfer function is identical to (b):

$$OL = K_p \cdot H$$

producing the same DC error, phase margin, and closed-loop bandwidth:

DC error $= 0$

Phase margin $= +90°$

Closed-loop -3dB BW $= 1{,}000$

Note: A lower Kd canceling the pole at -10 rather than the measurement pole produces the same 90 deg phase margin, but lower -3dB bandwidth at 10 rather than 1,000.

(e) [10 pts] Would additional integral control help to improve system performance in any way? Explain.

No! The DC error is already zero. Any additional integral control can only harm the phase margin.

2. **[30 pts]** Here we consider the dynamics of a coupled set of two ordinary differential equations in state variables $u(t)$ and $v(t)$ describing the interaction between two bacterial populations competing for growth, where the output population $u(t)$ is driven by a nutrient supply input $f(t)$:

$$\begin{aligned}\frac{du}{dt} &= -v(t)\\ \frac{dv}{dt} &= -u(t) + f(t).\end{aligned}$$

   (a) [10 pts] Find the transfer function $H(s) = u(s)\,/\,f(s)$ of the biosystem, and find the poles and zeros. Is the biosystem stable? Explain.

$$s u = -v$$

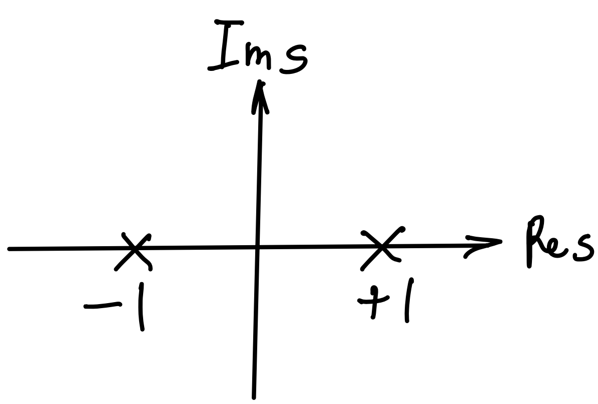
$$s v = -u + f$$

$$s^2 u = u - f$$

$$H(s) = \frac{u(s)}{f(s)} = \frac{1}{1 - s^2}$$

No zeros.

Two complementary real poles at: $s = \pm 1$



(b) [10 pts] Now consider proportional-derivative control $F(s) = K(1 - s)$ acting on the biosystem without measurement error, with $K > 1$. Find the phase margin and the closed-loop DC error. Explain what you observe.

$$OL = F(s) \cdot H(s)$$

$$= K(1-s) \cdot \frac{1}{1-s^2}$$

$$= K \frac{\cancel{1-s}}{(\cancel{1-s})(1+s)}$$

$$= \frac{K}{1+s}$$

Phase margin:

$PM = +90^\circ$ for K >> 1

$< +135^\circ$ for K > 1

DC error:

$$1 - CL(0) = \frac{1}{1+OL(0)} = \frac{1}{1+K}$$

(c) [10 pts] Now consider integral-derivative control $F(s) = K\left(\frac{1}{s} - s\right)$ acting on the biosystem without measurement error, with $K > 0$. Find the phase margin and the closed-loop DC error. Compare with (b), and explain.

$$OL = F(s) \cdot H(s)$$

$$= K \frac{1-s}{s} \cdot \frac{1}{1-s^2}$$

$$= K \frac{\cancel{1-s}}{s\cancel{(1-s)}(1+s)}$$

$$= \frac{K}{s(1+s)}$$

Phase margin:

$$PM = +0^\circ \quad \text{for } K >> 1$$

$$< +45^\circ \quad \text{for } K > 1$$

DC error:

$$1 - CL(0) = \frac{1}{1 + OL(0)} = \frac{1}{1+\infty} = 0$$

3. **[20 pts]** Short answer questions– let your imagination and creativity go loose!

(a) [10 pts] Give an example where derivative control is essential to improve on closed-loop feedback control of a biosystem subject to measurement error at high frequencies.

Any biosystem with second-order lowpass frequency response, for instance, deflection of a damped spring-mass system subjected to force. With just proportional control, the phase margin approaches zero as the gain is increased to lower DC error. Derivative control lifts the phase by 90 degrees at higher frequencies producing a phase margin at least 90 degrees yielding substantially greater stability.

(b) [10 pts] Give an example where integral control is essential to improve on closed-loop feedback control of a biosystem lacking gain at low frequencies.

Any biosystem with a highpass frequency response, for instance, velocity of a damped spring-mass system subjected to force. With just proportional control, the closed-loop gain is zero at zero frequency for any gain. Integral control boosts the magnitude by -20dB/dec at lower frequencies producing non-zero open-loop gain at zero frequency yielding substantially lower closed-loop DC error.