BEGN 221 Mathematical Methods in Bioengineering

Fall 2018

Midterm

NAME: SOLUTIONS

- Open book, open notes.
- 80 minutes limit (end of class).
- No communication other than with instructor and TAs.
- No computers or internet, except for access to posted class materials.

Table 1: Laplace and Fourier Transforms

$u(t)$	$U(s)$	$u(t)$	$U(j\omega)$
$\delta(t)$	1	$\delta(t)$	1
1	$\frac{1}{s}$	1	$\frac{1}{j\omega}$
e^{-at}	$\frac{1}{s+a}$	e^{-at} for $t \geq 0$; 0 otherwise	$\frac{1}{j\omega + a}$
$u(t-t_0);\ t_0 \geq 0$	$e^{-st_0}\,U(s)$	$u(t-t_0)$	$e^{-j\omega t_0}\,U(j\omega)$
$\frac{du}{dt}$	$s\,U(s) - u(0)$	$\frac{du}{dt}$	$j\omega\,U(j\omega)$
$\int_0^t u(t_0)\,dt_0$	$\frac{1}{s}\,U(s)$	$\int_{-\infty}^t u(t_0)\,dt_0$	$\frac{1}{j\omega}\,U(j\omega)$
$\int_0^t f(t_0)\,h(t-t_0)\,dt_0$	$H(s)\cdot F(s)$	$\int_{-\infty}^{+\infty} f(t_0)\,h(t-t_0)\,dt_0$	$H(j\omega)\cdot F(j\omega)$

Table 2: Green's Functions for Diffusion in 1-D

B.C. $x=0$	$x=L$	$G(x,t;x_0,t_0)$, $t > t_0$
—	—	$\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) = \frac{1}{\sqrt{4\pi D(t-t_0)}} \exp(-\frac{(x-x_0)^2}{4D(t-t_0)})$
$u(0,t)=0$	—	$\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) - \mathcal{N}(-x_0, \sqrt{2D(t-t_0)})$
$\frac{\partial u}{\partial x}(0,t)=0$	—	$\mathcal{N}(x_0, \sqrt{2D(t-t_0)}) + \mathcal{N}(-x_0, \sqrt{2D(t-t_0)})$
$u(0,t)=0$	$u(L,t)=0$	$\sum_{k=1}^{\infty} \frac{2}{L} \sin(\frac{k\pi}{L}x_0)\,\sin(\frac{k\pi}{L}x)\,\exp(-(\frac{k\pi}{L})^2 D(t-t_0))$
$u(0,t)=0$	$\frac{\partial u}{\partial x}(L,t)=0$	$\sum_{k=0}^{\infty} \frac{2}{L} \sin(\frac{(k+\frac{1}{2})\pi}{L}x_0)\,\sin(\frac{(k+\frac{1}{2})\pi}{L}x)\,\exp(-(\frac{(k+\frac{1}{2})\pi}{L})^2 D(t-t_0))$
$\frac{\partial u}{\partial x}(0,t)=0$	$u(L,t)=0$	$\sum_{k=0}^{\infty} \frac{2}{L} \cos(\frac{(k+\frac{1}{2})\pi}{L}x_0)\,\cos(\frac{(k+\frac{1}{2})\pi}{L}x)\,\exp(-(\frac{(k+\frac{1}{2})\pi}{L})^2 D(t-t_0))$
$\frac{\partial u}{\partial x}(0,t)=0$	$\frac{\partial u}{\partial x}(L,t)=0$	$\frac{1}{L} + \sum_{k=1}^{\infty} \frac{2}{L} \cos(\frac{k\pi}{L}x_0)\,\cos(\frac{k\pi}{L}x)\,\exp(-(\frac{k\pi}{L})^2 D(t-t_0))$

Problem 1 (20 points): Short answer problems. Provide brief explanations (no lengthy derivations!) for each problem.

1. (5 points): Specify the conditions on the matrix $\mathbf{A}$ so that the components c_n of the projection of any vector $\mathbf{g}$ onto the basis consisting of the eigenvectors $\mathbf{U}_n$ of $\mathbf{A}$ can be obtained by taking dot products as follows:

$$\mathbf{g} = \sum_{n=1}^{N} c_n \mathbf{U}_n \quad \text{where} \quad c_n = \frac{\mathbf{g} \cdot \mathbf{U}_n}{\mathbf{U}_n \cdot \mathbf{U}_n}$$

$\bar{\bar{A}}$ is symmetric, so that the eigenvectors $\vec{U}_n$ are orthogonal.

2. (5 points): Which type of homogeneous boundary conditions gives rise to a homogeneous solution that decays to zero over time?

Zero-value boundary conditions.

(Only zero-flux on all sides conserves initial energy.)

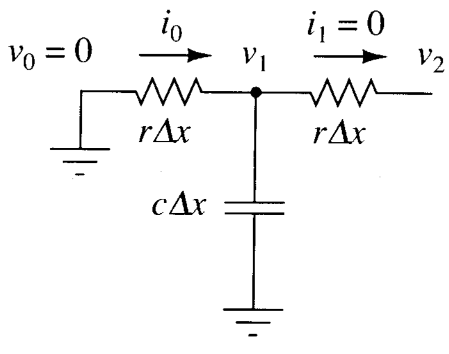
3. (5 points): What can you say about the Green's function $G(x, t; x_0, t_0)$ of a linear-time-invariant (LTI), linear-space-invariant (LSI) system?

$$G(x, t; x_0, t_0) = G(x - x_0, t - t_0; 0, 0)$$

4. (5 points): What is the inverse Laplace transform of the transfer function $H(s)$ of a LTI system?

The impulse response $h(t)$.

Problem 2 (30 points): Consider a two-segment lumped model of diffusion along a passive cable of length L, with line resistivity r and line capacitance c, and with zero-voltage boundary condition on the left end, and zero-current boundary condition on the right end, as shown below. The length of each of the two segments is $\Delta x = L/2$.



1. (10 points): Write the ordinary differential equation governing the dynamics of the voltage $v_1(t)$ at the center of the cable. What can you say about the voltage $v_2(t)$ on the right end?

$$c\Delta x \frac{dv_1}{dt} = \frac{0 - v_1}{r\Delta x}$$

$$\Rightarrow \frac{dv_1}{dt} = -\frac{1}{rc\Delta x^2} v_1 = -\frac{4D}{L^2} v_1$$

with $D = \frac{1}{rc}$ diffusivity

$v_2(t) = v_1(t)$ since $i_1 = 0$ (Ohm's law)

2. (10 points): Find the the solution to this ODE using your favorite method. Express your solution $v_1(t)$ in terms of the initial condition $v_1(0) = V_i$, cable length L, and diffusivity D.

e.g. Laplace : $v_1(t) \longrightarrow V_1(s)$

$$s\,V_1(s) - V_i = -\frac{4D}{L^2}\,V_1(s)$$

$$V_1(s) = V_i \cdot \frac{1}{s + \frac{4D}{L^2}}$$

$$v_1(t) = V_i \cdot e^{-\frac{4D}{L^2}t}$$

3. (10 points): Consider the current $i_0(t)$ flowing into the left end of the two-segment lumped model of the cable, in comparison to the current $i(0,t)$ flowing into the left end of the infinite-segment continuous model of the cable, with zero-voltage boundary conditions on the left end $u(0,t) = 0$, zero-current boundary conditions on the right end $i(L,t) = 0$, and uniform initial conditions $u(x,0) = V_i$. Initially, at $t = 0$, which of these two left-boundary currents $i_0(0)$ and $i(0,0)$ is larger, and why? *Hint*: the solution $u(x,t)$ for general x and t is not needed here.

$$i_0(0) = -\frac{1}{r\Delta x}\cdot v_1(0) = -\frac{2}{rL}\cdot V_i$$

$$i(0,0) = \lim_{\Delta x \to 0} -\frac{1}{r\Delta x}\cdot V_i = -\infty\ !$$

$i(0,0)$ is infinitely larger in magnitude than $i_0(0)$, because the voltage gradient in the infinite-segment model continuous limit becomes infinite:

$$i(0,0) = -\frac{1}{r}\frac{\partial v}{\partial x}(0,0) = -\infty \quad \text{since} \begin{cases} v(0,t) = 0 & \text{for } t=0 \\ v(x,0) = V_i & \text{for } x>0 \end{cases}$$

Problem 3 (50 points): Here we will investigate the effect of a bolus injection of a fixed amount of heat B on the temperature distribution $u(x,t)$ in a limb that is at serious risk of frostbite. The limb has length L with uniform mass density ρ, uniform heat capacitance c, and uniform heat conductance K. The initial temperature distribution is uniformly zero throughout the limb. A constant heat flux Φ_0 enters the limb from the rest of the body through its trunk connecting to the body on one side ($x = 0$). The other end ($x = L$) is thermally insulating, as is the entire surface of the limb. The heat bolus B is injected all at once at time $t = 0$, and concentrated in a single cross-section of the limb at distance $x = x_0$ from the trunk: $Q(x,t) = B\,\delta(t)\,\delta(x - x_0)\,/\,A$ where $\delta(\cdot)$ is the delta-Dirac function, and A is the cross-section area of the limb. You may assume that all diffusion in the limb is longitudinal in the x direction, and ignore any transversal effects along the other dimensions perpendicular to x.

1. (10 points): Write the partial differential equation governing the temperature $u(x,t)$ in the limb. Express initial and boundary conditions. *Hint*: check for consistency in the units!

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho} \quad \text{where} \begin{cases} Q(x,t) = \frac{B}{A}\,\delta(t)\,\delta(x-x_0) \\ D = \frac{K}{c\rho} \quad \text{DIFFUSIVITY} \end{cases}$$

$$-K\frac{\partial u}{\partial x}(0,t) = \Phi_0 \qquad \text{CONSTANT-FLUX B.C. @ } x=0$$

$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad \text{ZERO-FLUX B.C. @ } x=L$$

$$u(x,0) = 0 \qquad \text{ZERO I.C.}$$

2. (10 points): Now consider that the heat flux Φ_0 through the trunk is zero, so that the only source of heat entering the limb is the bolus. Reformulate this problem as an equivalent homogeneous problem, with homogeneous partial differential equation and boundary conditions, and with the effect of the bolus expressed as an initial condition.

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{ZERO-FLUX @ } x=0$$

$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad \text{ZERO-FLUX @ } x=L$$

$$u(x,0) = \frac{1}{c\rho}\cdot\frac{B}{A}\cdot\delta(x-x_0) \qquad \text{BOLUS I.C.}$$

3. (15 points): Still assuming $\Phi_0 = 0$, solve for the temperature distribution $u(x,t)$ using your method of choice. How does this solution relate to the Green's function for this system?

$$u(x,t) = \frac{1}{c\rho} \cdot \frac{B}{A} \cdot G(x,t;x_0,0)$$

since the source term is an impulse with magnitude $\frac{1}{c\rho}\frac{B}{A}$ @ $x = x_0$ and $t = 0$

From the Green's table (FLUX-FLUX B.C.):

$$u(x,t) = \frac{1}{c\rho}\frac{B}{A}\left(\frac{1}{L} + \underbrace{\frac{2}{L}\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{L}x_0\right)\cos\left(\frac{k\pi}{L}x\right)e^{-\left(\frac{k\pi}{L}\right)^2 Dt}}_{\to 0 \text{ for } t\to\infty}\right)$$

4. (5 points): Now consider the steady state response as the limit of your solution in Part 3 for $t \to \infty$. Interpret your result. How much bolus B is required to raise the temperate from the initial freezing to nominal body temperature T_b?

$$\lim_{t\to\infty} u(x,t) = \frac{1}{c\rho}\frac{B}{A}\frac{1}{L} = \frac{1}{c\rho}\frac{B}{V} \quad \text{where } V = A.L \text{ volume}$$

$\Rightarrow$ The heat bolus becomes uniformly distributed over the entire volume of the limb.

$\lim_{t\to\infty} u(x,t) = T_b$ for $B = c\rho V T_b$:

heat required to raise entire volume V by T_b.

5. (10 points): Now consider a non-zero positive heat flux Φ_0 entering from the rest of the body at the trunk $x = 0$, but without any bolus, $B = 0$. Express the balance in conservation of heat energy over the lumped volume of the limb to find approximately how much time is required for the entire limb to reach the nominal body temperature T_b from the initial freezing temperature.

Rate of change in heat energy over volume = Heat flux entering the left boundary surface

$$\frac{d}{dt}\left(c\rho \cancel{A} L \underbrace{u_p(t)}_{\text{lumped volume temperature}} \right) = \cancel{A}\cdot \Phi_0$$

$$\Rightarrow u_p(t) = \underbrace{\cancel{u_p(0)}}_{=0} + \frac{\Phi_0}{c\rho L}\cdot t$$

$$= T_b \text{ for } t = \frac{c\rho L}{\Phi_0}\cdot T_b$$

6. *Bonus* (+10 points): Find the exact solution $u(x,t)$ for this problem with non-zero Φ_0 and zero B, and compare with the uniform time-varying approximation $u_P(t)$ to the problem you obtained using the lumped model in Part 5.

The flux Φ_0 entering on the left is equivalent to a source with impulse of amplitude Φ_0 on the left:

$$Q(x,t) = \Phi_0 \cdot \delta(x)$$

$$\Rightarrow u(x,t) = \frac{1}{c\rho}\Phi_0 \int_0^t G(x,t;\,0,t_0)\,dt_0$$

$$= \frac{\Phi_0}{c\rho}\int_0^t \left(\frac{1}{L} + \frac{2}{L}\sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{L}x\right) e^{-\left(\frac{k\pi}{L}\right)^2 D(t-t_0)} \right) dt_0$$

$$= \underbrace{\frac{\Phi_0}{c\rho L}\cdot t}_{=u_p(t) \text{ from Part 5}} + \frac{2\Phi_0}{c\rho L}\sum_{k=1}^{\infty}\left(\frac{L}{k\pi}\right)^2 \frac{1}{D}\cos\left(\frac{k\pi}{L}x\right)\left(1 - e^{-\left(\frac{k\pi}{L}\right)^2 Dt}\right)$$